Topics on calculus in metric measure spaces

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TOPICS ON CALCULUS ON METRIC MEASURE SPACES

ANALYSE DANS LES ESPACES MÉTRIQUES MESURÉS

Thèse

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“为天地立心，为生民立命，为往圣继绝学，为万世开太平。”

[宋] 张载

To my parents

Abstract

This thesis concerns in some topics on calculus on metric measure spaces, in connection with optimal transport theory and curvature-dimension conditions. We study the continuity equations on metric measure spaces, in the viewpoint of continuous functionals on Sobolev spaces, and in the viewpoint of the duality with respect to absolutely continuous curves in the Wasserstein space. We study the Sobolev spaces of warped products of a real line and a metric measure space. We prove the 'Pythagoras theorem' for both cartesian products and warped products, and prove Sobolev-to-Lipschitz property for warped products under a certain curvature-dimension condition. We also prove the identification of p-weak gradients under curvature-dimension condition, without the doubling condition or local Poincaré inequality. At last, using the non-smooth Bakry-Émery theory on metric measure spaces, we obtain an improved Bochner inequality and propose a definition of N-Ricci tensor.

Key words: metric measure space, curvature-dimension condition, optimal transport, Sobolev space, Bakry-Émery theory, Ricci tensor.

Résumé

Cette thèse traite de plusieurs sujets d'analyse dans les espaces métriques mesurés, en lien avec le transport optimal et des conditions de courbure-dimension. Nous considérons en particulier les équations de continuité dans ces espaces, du point de vue de fonctionnelles continues sur les espaces de Sobolev, et du point de vue de la dualité avec les courbes absolument continues dans l'espace de Wasserstein. Sous une condition de courbure-dimension, mais sans condition de doublement de mesure ou d'inégalité de Poincaré, nous montrons également l'identification des p-gradients faibles. Nous étudions ensuite les espaces de Sobolev sur le produit tordu de l'ensemble des réels et d'un espace métrique mesuré. En particulier, nous montrons la propriété Sobolev-à-Lipschitz sous une certaine condition de courbure-dimension. Enfin, sous une telle condition et dans le cadre d'une théorie non-lisse de Bakry-Émery, nous obtenons une inégalité améliorée de Bochner et proposons une définition du N-tenseur de Ricci.

Mots-clés: espace métrique mesuré, condition de courbure-dimension, transport optimal, espace de Sobolev, théorie de Bakry-Émery, tenseur de Ricci.

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imperceptibly influence from you is more important. Beyond that, I cannot find a better reason for my deep love in mathematics and persistence in choosing mathematical research as my career.

Contents

Symbols

X	polish space
d	metric
$\mathfrak{m}$	measure
$\{\gamma_t\}_{t\in[0,1]}$	a curve
$\mathcal{P}(X)$	probability measures
$\mathcal{P}_p(X)$	probability measures with p-moment
$\mathcal{W}_2$	2-Wasserstein distance
$\mathrm{Lip}\, X$	set of Lipschitz functions
$\mathrm{Lip}(f)$	Lipschitz constant of f
$\mathrm{lip}(f)$	local Lipschitz constant of f
$S^p(X, \mathrm{d}, \mathfrak{m})$	p-Sobolev class
$W^{1,p}(X, \mathrm{d}, \mathfrak{m})$	Sobolev space
$\lvert \mathrm{D} f\rvert_p$	p-weak gradient of f
$X \times_w Y$	warped prodcut of X and Y with warping function w
$\mathrm{RCD}(K, \infty)$	metric measure spaces with Ricci curvature bounded below by K
$\mathrm{RCD}^*(K, N)$	metric measure spaces with lower Ricci bound and upper dimension bound

Résumé des Travaux

Dans une suite des travaux par Lott-Villani (voir [34, 35]) et Sturm (voir [40, 41]), la théorie de l'espace métrique mesuré avec la courbure de Ricci minorée (ou condition de courbure-dimension) a été construit. Plus récemment, les outils de calcul développés par Ambrosio-Gigli-Savaré (voir [8, 9], [23, 24]) nous offre des outils pour l'étudier des espaces métriques mesurés. Dans cette thèse, mon premier objectif est d'étudier la théorie des espaces de Sobolev dans les espaces métriques mesurés générals, le deuxième objectif est d'obtenir une meilleure compréhension de la structure différentielle des RCD espaces.

L'équation de continuité

Dans l'article d'Otto ([37]) et Benamou-Brenier ([17]), ils prouvent que les courbes absolument continues dans l'espace de Wasserstein peuvent être caractérisés par les équations de continuité. Dans Chapitre-2, nous étudions cette correspondance en cas de non-lisse, en utilisant le transport optimal et la théorie de l'espace de Sobolev.

Tout d'abord, nous donnons une définition de l'équation de continuité dans les espaces métriques mesurés généraux.

Définition 2.10. *Soit* $(X, \mathrm{d}, \mathfrak{m})$ *un espace métrique mesuré,* $\{\mu_t\}_{t\in[0,1]} \subset \mathcal{P}_2(X)$ *une courbe* $\mathcal{W}_2$*-continue avec compression bornée, et* $\{L_t\}_{t\in[0,1]}$ *une famille de fonctionelles sur* $S^2(X)$*.*

Nous disons que $\{\mu_t\}_t$ *résout l'équation de continuité*

$$\partial_t \mu_t = L_t, \tag{1}$$

si:

i) Pour chaque $f \in S^2(X)$ *l'application* $t \mapsto L_t(f)$ *est mesurable et l'application* $N : [0,1] \mapsto [0,\infty]$ *définie par*

$$\frac{1}{2}N_t^2 := \operatorname*{ess\,sup}_{f \in S^2(X)} L_t(f) - \frac{1}{2}\|f\|_{\mu_t}^2, \tag{2}$$

est $L^2(0,1)$ *intégrable, i.e. pour toute* f, $\frac{1}{2}N_t^2 \geq L_t(f) - \frac{1}{2}\|f\|_{\mu_t}^2$, *p.p.* t *et pour toute* $\bar{N}_t$ *satisfaisant cette propriété, on a* $N_t \leq \bar{N}_t$ *for p.p.* t.

ii) Pour chaque $f \in L^1 \cap S^2(X)$ *l'application* $t \mapsto \int f \,\mathrm{d}\mu_t$ *est absolument continue et*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f \,\mathrm{d}\mu_t = L_t(f),$$

pour p.p. $t \in [0,1]$.

Notre résultat principal affirme que pour une courbe $\{\mu_t\}_t$ avec compression bornée, l'équation de continuité caractérise $\mathcal{W}_2$-absolument continuité.

Théorème 2.11. *Soit* $\{\mu_t\}_t \subset \mathcal{P}(X)$ *une courbe* $\mathcal{W}_2$*-continue avec compression bornée. Alors les suivantes sont équivalentes.*

i) $\{\mu_t\}_t$ *est* $\mathcal{W}_2$*-absolument continue.*

ii) Il y a une famille de fonctionelles $\{L_t\}_{t\in[0,1]}$ *sur* $S^2(X)$ *t.q.* $\{\mu_t\}_t$ *résout l'équation de continuité (1).*

Enfin, on a

$$N_t = |\dot{\mu}_t|, \qquad \text{p.p. } t \in [0,1].$$

L'espace de Sobolev sur le produit tordu

Dans Chapitre-3 nous étudions les espaces de Sobolev sur produits cartésiens ainsi que des produits tordus de l'ensemble des réels et d'un espace métrique mesuré, qui sont utiles de la construction de nouveaux espaces.

On définit $\mathsf{BL}(X_w)$ comme le sous-ensemble de $L^2(X_w, \mathfrak{m}_w)$ des fonctions f t.q.

i) pour $\mathfrak{m}$-p.p. $x \in X$, on a $f^{(x)} \in W^{1,2}(\mathbb{R}, w_{\mathfrak{m}}\mathcal{L}^1)$,

ii) pour $w_{\mathfrak{m}}\mathcal{L}^1$-p.p. $t \in \mathbb{R}$, on a $f^{(t)} \in W^{1,2}(X)$,

iii)

$$|\mathrm{D}f|_w(t,x) := \sqrt{w_{\mathrm{d}}^{-2}(t)|\mathrm{D}f^{(t)}|_X^2(x) + |\mathrm{D}f^{(x)}|_{\mathbb{R}}^2(t)}$$

est $L^2(X_w, \mathfrak{m}_w)$ intégrable.

Ensuite, nous avons le théorème suivant qui caractérise l'espace de Sobolev.

Théorème *Soit $w_{\mathrm{d}}, w_{\mathfrak{m}}$ les fonctions continues t.q. $\{w_{\mathfrak{m}} = 0\} \subset I$ est discret et*

$$w_{\mathfrak{m}}(t) \le C \inf_{s: w_{\mathfrak{m}}(s)=0} |t-s|, \qquad \forall t \in \mathbb{R}.$$

Alors $W^{1,2}(X_w) = \mathsf{BL}(X_w)$ et pour chaque $f \in W^{1,2}(X_w) = \mathsf{BL}(X_w)$, on a

$$|\mathrm{D}f|_{X_w} = |\mathrm{D}f|_w \qquad \mathfrak{m}_w - \ p.p..$$

En particulier, nous montrons la propriété Sobolev-à-Lipschitz sous une certaine condition de courbure-dimension.

Théorème 3.30. *Soit $(X, \mathrm{d}, \mathfrak{m})$ un espace de $\mathrm{RCD}(K, \infty)$ doublement, $I \subset \mathbb{R}$ un intervalle fermé, $w_{\mathrm{d}}, w_{\mathfrak{m}} : I \to \mathbb{R}$ fonctions continues. Supposons que $w_{\mathfrak{m}}$ est positif dans l'intérieur du I.*

Alors, le produit tordu $(X_w, \mathrm{d}_w, \mathfrak{m}_w)$ a la propriété Sobolev-à-Lipschitz.

L'identification des p-gradients faibles

Il est connu que $|\mathrm{D}f|_p$, le p-gradient faible de $f \in S^p(X)$, peut être définie par relaxation des constantes de Lipschitz locales. Comme $p \mapsto \|\cdot\|_{L^p}$ est non-décroissante, par définition nous savons que $p \mapsto |\mathrm{D}f|_p$ est également non-décroissante.

Une question naturelle sur p-gradient faible est que si $|\mathrm{D}f|_p$ dépend de p ou non. C'est à dire: soit $\mathfrak{m} \in \mathcal{P}(X)$, $1 < p_1 < p_2$, peut-on dire que $|\mathrm{D}f|_{p_1} = |\mathrm{D}f|_{p_2}$ pour toute $f \in W^{1,p_2}(X, \mathrm{d}, \mathfrak{m})$? En plus, si $f \in W^{1,p_1}$ et $f, |\mathrm{D}f|_{p_1} \in L^{p_2}(X)$, peut-on dire que $f \in W^{1,p_2}(X, \mathrm{d}, \mathfrak{m})$?

En cas de variété riemannienne, il est connu que ces questions ont des réponses positives. Mais dans les espaces métriques mesurés généraux, les réponses sont négatifs (voir [36] pour un contre-exemple et aussi voir [6] pour un contre-exemple proposé par Koskela). Cependant, sous conditions de doublement et d'inégalité de Poincaré, $|\mathrm{D}f|_p$ est vraiment indépendant de p comme prouvé par Cheeger dans [20].

Notre résultat principal est le théorème suivant. Sous condition de $\mathrm{RCD}(K,\infty)$, mais sans condition de doublement de mesure ou d'inégalité de Poincaré, nous montrons également l'identification des p-gradients faibles.

Théorème 4.9 (L'identification des p-gradients faibles) *Soit* $p,q \in (1,\infty)$ *et* $f \in S^p_{\mathrm{loc}}(X)$ *t.q.* $|\mathrm{D}f|_p \in L^q_{\mathrm{loc}}(X)$. *Alors* $f \in S^q_{\mathrm{loc}}(X)$ *et*

$$|\mathrm{D}f|_q = |\mathrm{D}f|_p, \qquad \mathfrak{m} - p.p..$$

N-tenseur de Ricci

Soit M une variété riemannienne avec tenseur métriquee $\langle \cdot,\cdot \rangle : [TM]^2 \mapsto C^\infty(M)$. Nous avons formule de Bochner:

$$\Gamma_2(f) = \mathrm{Ricci}(\nabla f, \nabla f) + \|\mathrm{H}_f\|^2_{\mathrm{HS}}, \tag{3}$$

pour toute fonction lisee f, où $\|\mathrm{H}_f\|_{\mathrm{HS}}$ est la norme de Hilbert-Schmidt de la Hessienne $\mathrm{H}_f := \nabla \mathrm{d} f$ et l'opérateur Γ_2 est définie par

$$\Gamma_2(f) := \frac{1}{2} L\Gamma(f,f) - \Gamma(f, Lf), \qquad \Gamma(f,g) := \frac{1}{2}\big(L(fg) - fLg - gLf\big)$$

où $L = \Delta$ est l'opérateur de Laplace-Beltrami.

Soit M un $\mathrm{RCD}(K,\infty)$ espace. Nous avons formule de Bochner qui est montré (définie) par Gigli dans [23]:

$$\mathbf{\Gamma}_2(f) = \mathbf{Ricci}(\nabla f, \nabla f) + \|\mathrm{H}_f\|^2_{\mathrm{HS}}\, \mathfrak{m} \tag{4}$$

pour chaque $f \in \mathrm{TestF}(M)$.

Donc, nous voulons savoir si nous pouvons définir/ montrer une formule similaire dans les espaces métriques mesurés de condition de courbure-dimension $\mathrm{RCD}^*(K,N)$.

Tout d'abord, nous étudions la dimension d'un $\mathrm{RCD}^*(K,N)$ espace qui est considéré comme la dimension de la (L^∞-) module tangent $L^2(TM)$. Dans Chapitre-5 nous montrons que les dimensions des $\mathrm{RCD}^*(K,N)$ espaces sont majorée par N.

Proposition 5.12 *Soit* $M=(X,\mathrm{d},\mathfrak{m})$ *un* $\mathrm{RCD}^*(K,N)$ *espace, alors* $\dim M \leq N$. *En plus, si la dimension locale de la module tangent dans un ensemble Borel* E *est* N*, alors* $\mathrm{trH}_f(x)=\Delta f(x)$ $\mathfrak{m}$*-p.p.* $x\in E$ *pour* $f\in\mathrm{TestF}$.

Ensuite, nous obtenons une inégalité améliorée de Bochner.

Théorème 5.13. *Soit* $M=(X,\mathrm{d},\mathfrak{m})$ *un* $\mathrm{RCD}^*(K,N)$ *espace métrique mesuré, où* $N\geq\dim M$. *Alors, pour chaque* $f\in\mathrm{TestF}$, *on a*

$$\mathbf{\Gamma}_2(f) \quad\geq\quad \big(K|\mathrm{D}f|^2+\|\mathrm{H}_f\|^2_{\mathrm{HS}}+\frac{1}{N-\dim_{\mathrm{loc}}}(\mathrm{trH}_f-\Delta f)^2\big)\,\mathfrak{m}$$

où $\dim_{\mathrm{loc}}$ *est la dimension locale.*

Définition 5.15 (Ricci tensor) *On définit* $\mathbf{Ricci}_N$ *comme une application* $[H^{1,2}_H(TM)]^2\mapsto \mathrm{Meas}(M)$ *tel que pour* $X,Y\in\mathrm{TestV}(M)$

$$\mathbf{Ricci}_N(X,Y) \quad=\quad \mathbf{\Gamma}_2(X,Y)-\langle(\nabla X)^b,(\nabla Y)^b\rangle_{\mathrm{HS}}\,\mathfrak{m}-R_N(X,Y)\,\mathfrak{m}.$$

où

$$\mathbf{\Gamma}_2(X,Y):=\mathbf{\Delta}\frac{\langle X,Y\rangle}{2}+\Big(\frac{1}{2}\langle X,(\Delta_{\mathrm{H}}Y^b)^\sharp\rangle+\frac{1}{2}\langle Y,(\Delta_{\mathrm{H}}X^b)^\sharp\rangle\Big)\,\mathfrak{m},$$

et

$$R_N(X,Y):=\begin{cases}\dfrac{1}{N-\dim_{\mathrm{loc}}}\big(\mathrm{tr}(\nabla X)^b-\mathrm{div}X\big)\big(\mathrm{tr}(\nabla Y)^b-\mathrm{div}Y\big) & \dim_{\mathrm{loc}}<N,\\ 0 & \dim_{\mathrm{loc}}\geq N.\end{cases}$$

Comme un corollaire, nous pouvons écrire Théorème 5.13 et Théorème 3.6.7 de [23] comme:

Théorème 5.16 *Soit* M *est* $\mathrm{RCD}^*(K,N)$, *alors*

$$\mathbf{Ricci}_N(X,X)\geq K|X|^2\,\mathfrak{m},$$

et

$$\mathbf{\Gamma}_2(X,X) \quad \geq \quad \Big(\frac{(\mathrm{div}X)^2}{N} + \mathbf{Ricci}_N(X,X)\Big)\,\mathfrak{m}$$

pour chaque $X \in H^{1,2}_H(TM)$. *D'autre part, si* M *est* $\mathrm{RCD}(K',\infty)$, *et*

(1) $\mathrm{dim}M \leq N$

(2) $\mathrm{tr}(\nabla X)^b = \mathrm{div}X\ \mathfrak{m} - p.p.\ dans\ \{\mathrm{dim}_{\mathrm{loc}} = N\}, \forall X \in H^{1,2}_H(TM)$

(3) $\mathbf{Ricci}_N \geq K$

pour certains $K \in \mathbb{R}$, $N \in [1,+\infty]$, *alors* M *est* $\mathrm{RCD}^*(K,N)$.

Publications

Cette thèse reprend le contenu d'articles de journaux ou de prépublications suivants.

(1) N. Gigli and B.-X. Han, The continuity equation on metric measure spaces, Calc. Var. Partial Differential Equations, (2014).

(2) N. Gigli and B.-X. Han, Independence on p of weak upper gradients on RCD spaces. Prépublication, 2014.

(3) N. Gigli and B.-X. Han, Sobolev spaces on warped products. Prépublication, 2014.

(4) B.-X. Han, Ricci tensor on $\mathrm{RCD}^*(K,N)$ spaces. Prépublication, 2015.

Chapter 1

Introduction

In a sequence of seminal papers by Lott-Villani (see [34, 35]) and Sturm (see [40, 41]), the theory of metric measure spaces with synthetic lower Ricci curvature bounds (or curvature-dimension condition) was constructed. Thereafter, the research on metric measure space using optimal transport theory became more and more popular. More recently, the calculus tools developed by Ambrosio-Gigli-Savaré (see [8, 9], [23, 24]) offer us powerful analysis tools for the study of metric measure spaces. In this thesis, my first goal is to study the theory of Sobolev spaces on general metric measure spaces, the second goal is to obtain a better understanding of Sobolev calculus on the metric measure spaces with Ricci curvature bounded from below.

From the work of Otto ([37]) and Benamou-Brenier ([17]), we know that absolutely continuous curves in Wasserstein space can be described by continuity equations. In Chapter-2, we study this correspondence in non-smooth case, using the vocabulary of optimal transport and Sobolev space theory developed by Ambrosio-Gigli-Savaré in [8] and [9].

In Chapter-3 we study Sobolev space of cartesian products as well as warped products of the real line and metric measure spaces, which are useful ways of constructing new spaces. The 'Pythagoras type' formulas make it possible to compute the weak gradients in such products. Furthermore, we prove the Sobolev-to-Lipschitz property in a relevant class of spaces which includes the key cases of spheres and cones.

Considering the Sobolev space $W^{1,p}(X)$ with $p > 1$ where X is $\mathbb{R}^n$ or more generally a manifold, we know that the distributional derivative of any $f \in W^{1,p}(X)$ is well defined and independent of p. However, this is not the general case for non-smooth spaces. For example in [36] the authors construct an example showing that the p-weak gradient $|\mathrm{D}f|_p$ really depends on the choice of p, i.e. we may find a Sobolev function

$f \in W^{1,p}(X) \cap W^{1,q}(X)$ with $|\mathrm{D}f|_p \neq |\mathrm{D}f|_q$. This phenomenon makes it reasonable to ask what happens under additional assumptions on the space. From [20] we know the identification for metric measure spaces which are doubling and supporting a local Poincaré inequality. In particular the identification holds for $\mathrm{CD}(K,N)$ spaces. In Chapter-4, Theorem 4.9, we extend the identification result to $\mathrm{RCD}(K,\infty)$ spaces and partially answer a question posed in [2] about the BV and $W^{1,1}$ spaces.

In the last chapter, as an application of the Sobolev theory, differential structure of metric measure spaces and the Bakry-Émery theory on $\mathrm{RCD}(K,\infty)$ metric measure spaces, we study the dimension bound of $\mathrm{RCD}^*(K,N)$ spaces. Furthermore, we prove an improved Bochner inequality and give a definition of finite dimensional Ricci tensor on non-smooth metric measure spaces.

1.1 Basic notions

Let (X,d) be a complete metric space. We denote the space of continuous curves on X as $\mathrm{C}([0,1],X)$ and denote the space of absolutely continuous curves as $\mathrm{AC}([0,1],X)$. We denote the space of geodesics as Geo(X). For $t \in [0,1]$, the evaluation map e_t : $\mathrm{C}([0,1],X) \mapsto X$ is given by

$$e_t(\gamma) := \gamma_t, \quad \forall \gamma \in \mathrm{C}([0,1],X).$$

For $t,s \in [0,1]$ the map restr_t^s from $C([0,1],X)$ to itself is given by

$$(\mathrm{restr}_t^s \gamma)_r := \gamma_{t+r(s-t)}, \qquad \forall \gamma \in C([0,1],X).$$

The length of $\gamma \in \mathrm{AC}([0,1],X)$ is computed by $\int_0^1 |\dot{\gamma}_t| \,\mathrm{d}t$ where $|\dot{\gamma}_t|$ is the metric speed of γ. Let $p>1$, the space of p-absolutely continuous curves is defined as the space of $\gamma \in \mathrm{AC}([0,1],X)$ such that $\int_0^1 |\dot{\gamma}_t|^p \,\mathrm{d}t < +\infty$, and is denoted as $\mathrm{AC}^p([0,1],X)$.

In this thesis, we are not only interested in metric structures, but also in the interaction between metrics and measures. For the metric measure space $(X,\mathrm{d},\mathfrak{m})$, basic assumptions used in this thesis are:

Assumption 1.1. *The metric measure space* $(X,\mathrm{d},\mathfrak{m})$ *satisfies:*

- (X,d) *is a complete and separable geodesic metric space;*
- $\mathfrak{m}$ *is a* σ*-finite Borel measure with respect to* d*;*
- $\mathfrak{m}$ *is finite on bounded sets;*

- $\operatorname{supp} \mathfrak{m} = X$.

Moreover, for brevity we will not distinguish $X, (X, \mathrm{d})$ or $(X, \mathrm{d}, \mathfrak{m})$ when no ambiguities exist. For example, we can write $W^{1,2}(X)$ instead of $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ (See next section).

Let $\mathcal{P}(X)$ be the space of probability measures and $p \in [1, \infty)$. We define $\mathcal{P}_p(X)$ as its subset consisting of measures with finite p-moment, i.e. $\mu \in \mathcal{P}_p(X)$ if $\mu \in \mathcal{P}(X)$ and $\int \mathrm{d}^p(x, x_0)\,\mathrm{d}\mu(x) < +\infty$ for some $x_0 \in X$. In optimal transport theory, we know $\mathcal{P}_p(X)$ equipped with the p-Wasserstein distance $\mathcal{W}_p$ is a complete and separable geodesic space. If Y is another metric space and $f : X \mapsto Y$ a Borel map, we denote $f_\sharp \mu \in \mathcal{P}(Y)$ as the push-forward measure (or image measure) of $\mu \in \mathcal{P}(X)$, which is also a probability measure such that $f_\sharp \mu(B) = \mu(f^{-1}(B))$ for any Borel set $B \subset Y$.

We have a correspondence between geodesics (absolutely continuous curves) $\{\mu_t\}_t$ in $(\mathcal{P}_p(X), \mathcal{W}_p)$ and probability measures $\mathcal{P}_2(\mathrm{Geo(X)})$ ($\mathcal{P}_p(\mathrm{AC}([0,1], X))$ respectively), i.e. $\mu_t = (e_t)_\sharp \pi$ with $\pi \in \mathcal{P}(\mathrm{Geo(X)})$ ($\mathcal{P}(\mathrm{AC}^p([0,1], X))$ respectively. We call such π a lifting of $\{\mu_t\}_t)$ such that π has the minimal energy $\int \int_0^1 |\dot\gamma_t|^p \,\mathrm{d}t\,\mathrm{d}\pi(\gamma)$. More details can be found in [33].

We use $\mathrm{CD}(K, \infty), \mathrm{CD}(K, N), \mathrm{CD}^*(K, N)$ $(\mathrm{RCD}(K, \infty), \mathrm{RCD}(K, N), \mathrm{RCD}^*(K, N))$ to denote the curvature-dimension conditions, where K means lower Ricci bound and N means upper dimension bound (or $N = \infty$ for the dimension free case). In general, the pair (K, N) should be seen as a unity, there makes no sense to understand them separately. The letter R in the notion of $\mathrm{RCD}(K, \infty), \mathrm{RCD}(K, N)$ and $\mathrm{RCD}^*(K, N)$ means 'Riemannian like' which are the metric measure spaces which are also infinitesimally Hilbertian. All the precise definitions can be found in Chapter-3 or in the references [3], [12] and [43].

1.2 Sobolev spaces and continuity equation

Let (X, d) be a metric space. For $f : X \mapsto \mathbb{R}$, the local Lipschitz constant $\mathrm{lip}(f) : X \mapsto [0, \infty]$ is defined as

$$\mathrm{lip}(f)(x) := \varlimsup_{y \to x} \frac{|f(y) - f(x)|}{d(x, y)}$$

if x is not isolated, and 0 otherwise. The Lipschitz constant is defined as

$$\mathrm{Lip}(f) := \sup_{x \neq y} \frac{|f(y) - f(x)|}{\mathrm{d}(x, y)}.$$

If (X, d) is a geodesic metric space, we have $\mathrm{Lip}(f) = \sup_x \mathrm{lip}(f)(x)$ for Lipschitz functions.

Now we introduce the Sobolev space on metric measure spaces, the first definition of Sobolev class is based on a relaxation procedure. We say that $f \in W^{1,p}(X), p > 1$ if we can find a sequence of Lipschitz functions $\{f_n\} \subset L^p(X)$ such that $f_n \to f$ in L^p and $\mathrm{lip}(f_n) \to G$ for some $G \in L^p$.

Another equivalent definition (see Theorem 7.4 in [6]) of the Sobolev space is as the following. A Borel function $f : X \mapsto \mathbb{R}$ belongs to the Sobolev class $S^p(X)$ if there exists a function $0 \le G \in L^p(X)$, called p-weak upper gradient such that

$$\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi(\gamma) \le \int \int_0^1 G(\gamma_s)|\dot{\gamma}_s| \,\mathrm{d}s \,\mathrm{d}\pi(\gamma),$$

for all q-test plans π. A q-test plans π is probability measure concentrated on $\mathcal{P}(\mathrm{AC}^q([0,1], X))$, $\frac{1}{p} + \frac{1}{q} = 1$ satisfying

$$\int_0^1 \int |\dot{\gamma}_t|^q \,\mathrm{d}\pi(\gamma) \,\mathrm{d}t < +\infty,$$

and having bounded compression, i.e. there exists $C > 0$ such that

$$(e_t)_\sharp \pi \le C\mathfrak{m}, \quad \forall t \in [0,1].$$

From [8], we know that there exists a minimal function G in the $\mathfrak{m}$-a.e. sense among all the p-weak upper gradients of f. We denote such minimal function by $|\mathrm{D}f|_p$ and call it p-minimal weak upper gradient or p-weak gradient for simplicity. Then the Sobolev space $W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ is defined as $W^{1,p}(X, \mathrm{d}, \mathfrak{m}) := S^p(X, \mathrm{d}, \mathfrak{m}) \cap L^p(X, \mathfrak{m})$ endowed with the norm

$$\|f\|^p_{W^{1,p}(X,\mathrm{d},\mathfrak{m})} := \|f\|^p_{L^p(X,\mathfrak{m})} + \||\mathrm{D}f|_p\|^p_{L^p(X,\mathfrak{m})}.$$

It can be seen from the definition that the $W^{1,p}$ norm $\|\cdot\|_{W^{1,p}}$ is a lower semi-continuous functional on $L^p(X, \mathfrak{m})$ with respect to L^1 convergence. This lower semi-continuity plays an important role in some of our topics later.

It is known (see [20] and [8]) that $W^{1,p}(X)$ is a Banach space. In general $W^{1,2}(X)$ is not a Hilbert space, For instance, in the case of Finsler manifolds, $W^{1,2}(X)$ is a Hilbert space if and only if X is a Riemannian manifold. We say that $(X, \mathrm{d}, \mathfrak{m})$ is an infinitesimally Hilbertian space if $W^{1,2}(X)$ is an Hilbert space.

In the followings of this section, we study the absolutely continuous curves in $(\mathcal{P}_2(X), \mathcal{W}_2)$ using the Sobolev space $W^{1,2}(X)$. We will not assume that $W^{1,2}(X)$ is a Hilbert space at this moment.

From the article by Otto (see [37]), and the work of Benamou-Brenier (see [17]), we know that absolutely continuous curves of measures $\{\mu_t\}_{t\in[0,1]}$ w.r.t. the 2-Wasserstein distance $\mathcal{W}_2$ on $\mathbb{R}^d$ can be interpreted as solutions of the continuity equation

$$\partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0, \tag{1.1}$$

where the vector fields v_t should be considered as the 'velocity' of the moving mass μ_t and, for curves with square-integrable speed, satisfy

$$\int_0^1\int |v_t|^2\,\mathrm{d}\mu_t\,\mathrm{d}t < \infty. \tag{1.2}$$

This intuition has been made rigorous by Ambrosio, Gigli and Savaré in [5], where it has been used to develop a rigorous first order calculus on the space $(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_2)$, with particular focus on the study of gradient flows.

Heuristically speaking, the continuity equation describes the link existing between the 'vertical derivative' $\partial_t\mu_t$ and the 'horizontal displacement' v_t. In this sense it provides the crucial link between analysis made on the L^p spaces, where the distance is measured 'vertically', and the one based on optimal transportation, where distances are measured by 'horizontal' displacement. This is indeed the heart of the crucial substitution made by Otto in [37] who, to define the metric tensor g_μ on the space $(\mathcal{P}_2(\mathbb{R}^d),\mathcal{W}_2)$ at a measure $\mu=\rho\mathcal{L}^d$ considers a 'vertical' variation $\delta\rho$ such that $\int\delta\rho\,\mathrm{d}\mathcal{L}^d=0$, then looks for solutions of

$$\delta\rho = -\nabla\cdot(\nabla\varphi\rho), \tag{1.3}$$

and finally defines

$$g_\mu(\delta\rho,\delta\rho) := \int|\nabla\varphi|^2\,\mathrm{d}\mu. \tag{1.4}$$

The substitution (1.3) is then another way of thinking at the continuity equation, while the definition (1.4) corresponds to the integrability requirement (1.2).

On Euclidean spaces it often happens that the continuity equation can be written in the form:

$$\partial_t\mu_t + \nabla\cdot(\nabla\varphi_t\mu_t) = 0,$$

for some functions φ_t, i.e. the vector fields v_t can be represented as gradients of functions. In particular, we have two examples.

The first one is the heat flow, which can be seen as the gradient flow of the relative entropy:

$$\partial_t\mu_t + \nabla\cdot(\nabla(-\log(\rho_t))\mu_t) = 0,$$

where $\mu_t=\rho_t\mathfrak{m}$.

The second example is the geodesic in Wasserstein space:

$$\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0,$$

with $\phi_t = -Q_{1-t}(-\varphi^c)$ for the geodesics, where φ is a Kantorovich with respect to the pair (μ_0, μ_1). This result can also be seen as a corollary of the famous Brenier's theorem.

Then we want to know whether the above discussions have non-smooth counterparts. We have the following questions to answer:

a) Is it possible to formulate the continuity equation on general metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$?

b) Do solutions of the continuity equation completely characterize absolutely continuous curves $\{\mu_t\}_t \subset \mathcal{P}(X)$ with square-integrable speed w.r.t. $\mathcal{W}_2$ and such that $\mu_t \leq C\mathfrak{m}$ for every $t \in [0, 1]$ and some $C > 0$?

c) In which case can the continuity equation be written in gradient form, i.e. $v_t = \nabla \varphi_t$ for some Sobolev functions $\{\varphi_t\}_t$?

To answer these questions, it is natural to consider the interaction between the Sobolev space $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ and absolutely continuous curves. To explain the ideas and motivations, we go back to the case when $X = \mathbb{R}^d$. Let $\{\mu\}_t \subset \mathcal{P}(\mathbb{R}^d)$ be an absolutely continuous curve. We define $\mathrm{Tan}_{\mu_t} = \overline{\{\nabla \varphi, \varphi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\mu_t)}$. Now we let $D \subset C_c^\infty(\mathbb{R}^d)$ be a countable set such that $\{\nabla \varphi : \varphi \in D\}$ is dense in Tan_{μ_t} for every $t \in [0, 1]$. Then one can show that (see [3]) there exists a set $A \in [0, 1]$ of full Lebesgue measure such that $t \mapsto \int \varphi \, \mathrm{d}\mu_t$ is differentiable at $t \in A$ for every $\varphi \in D$ and the metric derivative $|\dot{\mu}_t|$ exists. Then for each $t \in A$ we define the functional $L_t : \{\nabla \varphi : \varphi \in D\} \mapsto \mathbb{R}$ as:

$$\nabla \varphi \mapsto L_t(\nabla \varphi) := \frac{\mathrm{d}}{\mathrm{d}t} \int \varphi \, \mathrm{d}\mu_t.$$

With some work (see [33]) and [3]) one can also check that

$$|L_t(\nabla \varphi)| \leq \|\nabla \varphi\|_{L^2(\mu_t)} |\dot{\mu}_t|$$

and therefore L_t can be extended to a continuous linear functional on Tan_{μ_t}. Thus by the Riesz representation theorem there exists a vector field $v_t \in \mathrm{Tan}_{\mu_t}$ such that

$$L_t(\nabla \varphi) = \int \langle \nabla \varphi, v_t \rangle \, \mathrm{d}\mu_t, \qquad \forall \varphi \in D,$$

and $\|L_t\|^* = \|v_t\| \leq |\dot{\mu}_t|$.

Conversely, using the Kantorovich dual formula in optimal transport theory and a argument by Kuwada in [32], we can prove absolutely continuity from a continuity equation.

Following the ideas above, in Chapter-2 we answer the questions.

Answer to a): Functionals $\{L_t\}_t$ can be defined in the same way as in $\mathbb{R}^d$, but we might be not able to find a dense set D as above, therefore $\{L_t\}_t$ should only be seen as a family of functionals. At this point, let us recall that $\{\mu_t\}_{t\in[0,1]} \subset \mathcal{P}_2(X)$ has bounded compression if $\mu_t \leq C\mathfrak{m}$ $\forall t \in [0,1]$ for some $C \in \mathbb{R}$.

Definition 2.10 *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space, $\{\mu_t\}_{t\in[0,1]} \subset \mathcal{P}_2(X)$ a $\mathcal{W}_2$-continuous curve with bounded compression, and $\{L_t\}_{t\in[0,1]}$ a family of maps from $S^2(X)$ to $\mathbb{R}$.*

We say that $\{\mu_t\}_t$ solves the continuity equation

$$\partial_t \mu_t = L_t, \tag{1.5}$$

provided:

i) for every $f \in S^2(X)$ the map $t \mapsto L_t(f)$ is measurable and the map $N : [0,1] \mapsto [0,\infty]$ defined by

$$\frac{1}{2}N_t^2 := \operatorname*{ess\,sup}_{f\in S^2(X)} L_t(f) - \frac{1}{2}\|f\|_{\mu_t}^2, \tag{1.6}$$

belongs to $L^2(0,1)$, i.e. for any f, $\frac{1}{2}N_t^2 \geq L_t(f) - \frac{1}{2}\|f\|_{\mu_t}^2$ for a.e. t and for any other $\bar{N}_t$ satisfying this property, we have: $N_t \leq \bar{N}_t$ for a.e. t.

ii) for every $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f \,\mathrm{d}\mu_t$ is absolutely continuous and the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f \,\mathrm{d}\mu_t = L_t(f),$$

holds for a.e. t.

Remark 1.2. It can be seen that ii) in the definition above means the functional L_t is continuous.

Our main result asserts that for curves $\{\mu_t\}_t$ with bounded compression, the continuity equation characterizes 2-absolute continuity.

Answer to b): Here we replace the correspondence between vector fields v_t and absolutely continuous curves $\{\mu_t\}_t$ by L_t and $\{\mu_t\}_t$ due to lack of Riesz representation theorem.

Theorem 2.11 *Let $\{\mu_t\}_t \subset \mathcal{P}(X)$ be a $\mathcal{W}_2$-continuous curve with bounded compression. Then the followings are equivalent.*

i) $\{\mu_t\}_t$ *is* $\mathcal{W}_2$*-absolutely continuous curve.*

ii) There is a family of maps $\{L_t\}_{t\in[0,1]}$ *from* $S^2(X)$ *to* $\mathbb{R}$ *such that* $\{\mu_t\}_t$ *solves the continuity equation* (1.5)*.*

Finally, we have

$$N_t = |\dot{\mu}_t|, \qquad a.e.\ t \in [0,1].$$

It can be seen in the following way that this continuity equation, or the family of maps $\{L_t\}_{t\in[0,1]}$ is nothing but the 'optimal lift' of the corresponding absolutely continuous curve.

As we have mentioned in the last subsection, the absolutely continuous curves can be characterized by probability measures on the space of curves as follows:

(Superposition principle, [33]) Let (X, d) be a complete and separable metric space, and $\{\mu_t\}_{t\in[0,1]} \in \mathrm{AC}^2([0,1], \mathcal{P}_2)$. Then there exists a measure $\pi \in \mathcal{P}(\mathrm{C}([0,1], X))$ concentrated on $\mathrm{AC}^2([0,1], X)$ such that:

$$\begin{aligned} (e_t)_\sharp \pi &= \mu_t, \qquad &\forall t \in [0,1] \\ \int |\dot{\gamma}_t|^2 \,\mathrm{d}\pi(\gamma) &= |\dot{\mu}_t|^2, \qquad &\text{a.e. } t. \end{aligned}$$

Here, as we know the inequality

$$\int |\dot{\gamma}_t|^2 \,\mathrm{d}\pi(\gamma) \geq |\dot{\mu}_t|^2, \qquad \text{a.e. } t$$

holds for any π with $(e_t)_\sharp \pi = \mu_t$, the superposition principle tells us that there exists a plan $\pi \in \mathcal{P}(\mathrm{AC}([0,1], X))$ whose 'energy' is minimal. Using the language of Definition 2.10 and Theorem 2.11 we know this superposition plan corresponds to the continuity equation $\{L_t\}_t$ with $N_t = |\dot{\mu}_t|$.

Answer to c): In the general case, $\mathrm{D}f(\nabla g)$ may not be certainly defined. However, by a variational procedure we have the functions $\mathrm{D}^{\pm} f(\nabla g) : X \mapsto \mathbb{R}$ which are $\mathfrak{m}$-a.e. well defined by

$$\begin{aligned} \mathrm{D}^+ f(\nabla g) &:= \lim_{\varepsilon \downarrow 0} \frac{|\mathrm{D}(g + \varepsilon f)|^2 - |\mathrm{D}g|^2}{2\varepsilon}, \\ \mathrm{D}^- f(\nabla g) &:= \lim_{\varepsilon \uparrow 0} \frac{|\mathrm{D}(g + \varepsilon f)|^2 - |\mathrm{D}g|^2}{2\varepsilon}. \end{aligned}$$

It can be seen from the convexity of the map $\epsilon \mapsto |\mathrm{D}(g+\epsilon f)|^2$ that $\mathrm{D}^- f(\nabla g) \le \mathrm{D}^+ f(\nabla g)$. In case $(X, \mathrm{d}, \mathfrak{m})$ is a Riemannian manifold, the $\mathrm{D}^+ f(\nabla g)$ and $\mathrm{D}^- f(\nabla g)$ are coincide and are equal to $\mathrm{D}f(\nabla g)$.

Therefore, we say that a continuity equation is in gradient form if it satisfies

$$\int \mathrm{D}^- f(\nabla \varphi_t)\,\mathrm{d}\mu_t \le \frac{\mathrm{d}}{\mathrm{d}t}\int f\,\mathrm{d}\mu_t \le \int \mathrm{D}^+ f(\nabla \varphi_t)\,\mathrm{d}\mu_t, \qquad \text{a.e. } t$$

for suitable $\{\varphi_t\}$.

It can be seen (in $\mathbb{R}^d$) that the continuity equation can be written in a gradient form with some vector fields $\nabla\varphi_t$ if and only if the equality $L_t(\nabla\varphi_t) = \|\nabla\varphi_t\|_{L^2(\mu_t)}|\dot{\mu}_t|$ holds. In non-smooth case, this corresponds to the following definition which is proposed by De Giorgi as another type of gradient flow.

Definition 1.3 (Plans representing gradients)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space, $g \in W^{1,2}(X)$ and π a test plan. We say that π represents the gradient of g if it is a test plan and

$$\varliminf_{t\downarrow 0}\int \frac{g(\gamma_t)-g(\gamma_0)}{t}\,\mathrm{d}\pi(\gamma) \ge \frac{1}{2}\int |\mathrm{D}g|^2(\gamma_0)\,\mathrm{d}\pi(\gamma) + \frac{1}{2}\varlimsup_{t\downarrow 0}\frac{1}{t}\iint_0^t |\dot{\gamma}_s|^2\,\mathrm{d}s\,\pi(\gamma).$$

Then we have:

Theorem 2.23 *Let $\{\mu_t\} \in \mathrm{AC}^p([0,1], \mathcal{P}_2(X)$ be a curve with bounded compression, $(t,x) \mapsto \phi_t(x)$ a Borel map such that $\phi_t \in S^2(X)$ for a.e. $t \in [0,1]$ and π a lifting of $\{\mu_t\}_t$. Then*

i) Assume that $(\mathrm{restr}_t^1)_\sharp\pi$ represents the gradient of $(1-t)\phi_t$ for a.e. $t \in [0,1]$. Then $\{\mu_t\}_t$ solves the continuity equation (2.45).

ii) Assume that $S^2(X)$ is separable and that $\{\mu_t\}_t$ solves the continuity equation (2.45)*. Then $(\mathrm{restr}_t^1)_\sharp\pi$ represents the gradient of $(1-t)\phi_t$ for a.e. $t \in [0,1]$.*

Remark 1.4. From the results in [1] we know that $S^2(X)$ is separable if $(X, \mathrm{d}, \mathfrak{m})$ is doubling and $\mathfrak{m}$ finite on bounded sets (see also Proposition-2.4 for more details).

1.3 Sobolev space in warped products

The construction of new metric measure spaces from old ones is an important subject of metric geometry. One useful method is to construct cartesian product or more generally warped product space based on given ones.

We know that some important geometry results are related to the curvature-dimension of (warped) product spaces of an interval and metric measure spaces, for example cones and spheres (see [31] for the proof of the maximal diameter theorem). To use the calculus tools on metric measure spaces, it is useful to study the Sobolev space of the (warped) product spaces of a real line and a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ first. In particular, we want to know the relationship between the Sobolev spaces of (warped) product spaces and the Sobolev spaces of $(X, \mathrm{d}, \mathfrak{m})$.

We start from the case of the cartesian product space (or product space for abbreviation), which is basic but important. Here, we recall that the product space of two metric measure spaces $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ is defined as a metric measure space $(X \times Y, \mathrm{d}_c, \mathfrak{m}_c)$, for the distance $\mathrm{d}_c := \mathrm{d}_X \times \mathrm{d}_Y$ and the measure $\mathfrak{m}_c := \mathfrak{m}_X \times \mathfrak{m}_Y$. We know that $\mathrm{d}_c = \mathrm{d}_X \times \mathrm{d}_Y$ can be equivalently defined in the following two ways: The first one is the 'Pythagoras formula'

$$\mathrm{d}_c((x_1, y_1), (x_2, y_2)) = \sqrt{\mathrm{d}_X^2(x_1, x_2) + \mathrm{d}_Y^2(y_1, y_2)},$$

for any pairs $(x_1, y_1), (x_2, y_2) \in X \times Y$.

The second one is to minimize the length of curves:

$$\mathrm{d}_c(A, B) = \inf\{l[\gamma] : \gamma \text{ is an absolutely continuous curve from } A \text{ to } B\},$$

where the $l[\gamma]$ is the length of γ defined as

$$l[\gamma] = \int_0^1 \sqrt{|\dot{\gamma}_X|^2(t) + |\dot{\gamma}_Y|^2(t)}\, \mathrm{d}t,$$

where $|\dot{\gamma}_X|$ and $|\dot{\gamma}_Y|$ represent the speed of the curves γ_X, γ_Y respectively.

Now we switch to the study of Sobolev space on cartesian products. Inspired by the case in $\mathbb{R}^d$, we expect to prove the following 'Pythagoras formula':

$$|\mathrm{D}f|^2(t, x) = |\mathrm{D}f^{(x)}|_{\mathbb{R}}^2(t, x) + |\mathrm{D}f^{(t)}|_X^2(t, x), \qquad \text{a.e. } (t, x) \in \mathbb{R} \times X, \tag{1.7}$$

for $f \in W^{1,2}(\mathbb{R} \times X)$, where $|\mathrm{D}f^{(t)}|_X(t, x)$ and $|\mathrm{D}f^{(x)}|_{\mathbb{R}}(t, x)$ represent the weak gradient of $f^{(t)} := f(t, \cdot)$ and $f^{(x)} := f(\cdot, x)$ at $(t, x) \in \mathbb{R} \times X$ respectively. In [9] the authors give the affirmative answer provided $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{RCD}(K, \infty)$ space.

In Chapter-3 we study the Sobolev spaces $W^{1,2}$ of the cartesian products of the real line and metric measure spaces. Unlike the work in [9], we will be able to prove that the natural 'Pythagoras type' formula holds in full generality. The rigorous description is as the following:

Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space. We put $X_c := \mathbb{R} \times X$, $\mathfrak{m}_c := \mathcal{L}^1 \times \mathfrak{m}$ and $\mathrm{d}_c\big((t,x),(s,y)\big) := \sqrt{|s-t|^2 + \mathrm{d}^2(x,y)}$. We define the Beppo Levi space $\mathsf{BL}(X_c)$:

Definition 1.5. The space $\mathsf{BL}(X_c) \subset L^2(X_c, \mathfrak{m}_c)$ is the space of functions $f \in L^2(X_c, \mathfrak{m}_c)$ such that

i) $f^{(x)} \in W^{1,2}(\mathbb{R})$ for $\mathfrak{m}$-a.e. x,

ii) $f^{(t)} \in W^{1,2}(X)$ for $\mathcal{L}^1$-a.e. t,

iii) the function

$$|\mathrm{D}f|_c(t,x) := \sqrt{|\mathrm{D}f^{(t)}|_X^2(x) + |\mathrm{D}f^{(x)}|_{\mathbb{R}}^2(t)}$$

belongs to $L^2(X_c, \mathfrak{m}_c)$.

Then we have the following result.

Theorem 3.18 *We have $W^{1,2}(X_c) = \mathsf{BL}(X_c)$ as sets and for every $f \in W^{1,2}(X_c) = \mathsf{BL}(X_c)$ the identity*

$$|\mathrm{D}f|_{X_c} = |\mathrm{D}f|_c \qquad \mathcal{L}^1 \times \mathfrak{m} - \ a.e.,$$

holds.

Remark 1.6. It is proved in [9], the inequality

$$|\mathrm{D}f|_{X_c} \geq |\mathrm{D}f|_c \qquad \mathcal{L}^1 \times \mathfrak{m} - \ \text{a.e.},$$

holds for all Sobolev functions, the opposite inequality holds under on the $\mathrm{RCD}(K,\infty)$ assumption of $(X, \mathrm{d}, \mathfrak{m})$.

Our strategy to prove this theorem is as the following. Firstly we prove that a family of Sobolev functions $\mathcal{A}$ are dense in energy in $\mathsf{BL}(X_c)$, i.e. for any $f \in \mathsf{BL}(X_c)$ we can find a sequence of functions $\{f_n\}_n \subset \mathcal{A}$ such that $f_n \to f$ in L^2 and

$$\int |\mathrm{D}f_n|_c^2 \,\mathrm{d}\mathfrak{m} \to \int |\mathrm{D}f|_c^2 \,\mathrm{d}\mathfrak{m}.$$

Next, we prove the equality $|\mathrm{D}f|_{X_c} = |\mathrm{D}f|_c$ for $f \in \mathcal{A}$. At last, combining the lower semi-continuity and the inequality in Remark 1.6 we prove the theorem.

Based on the results on cartesian products, we turn to warped product spaces, which is a generalization of the cartesian products. Let $w_\mathrm{d}, w_\mathfrak{m}$ be continuous functions such that $\{w_\mathrm{d}(t) = 0\} \subset \{w_\mathfrak{m}(t) = 0\}$. One can construct the warped product $(X_w, \mathrm{d}_w, \mathfrak{m}_w)$ in a pure intrinsic way (see Definition 3.11). In the case when X is a Riemannian manifold M^n equipped with a metric tensor g, the usual warped product with respect to the warping function w is a Riemannian manifold, with metric tensor $\mathrm{d}_{w_{\mathrm{d}}} = \mathrm{d}t^2 + w^2(t)g$ and measure $w_{\mathfrak{m}} = w^n(t)$. In general, let X be general metric space. We can also define a complete metric on it by the following procedure. More details can be found in Chapter-3.

Let (X, d) be a complete geodesic space, and w be a continuous non negative function. Let $\gamma = (\gamma_{\mathbb{R}}, \gamma_X) : [0, 1] \mapsto \mathbb{R} \times X$ be a curve where $\gamma_{\mathbb{R}}$ and γ_X are absolutely continuous. Then the w-length of γ is defined in the following way:

$$l_w[\gamma] := \lim_{\tau} \sum_{i=1}^{n} \sqrt{|\gamma_{\mathbb{R}}(t_{i-1}), \gamma_{\mathbb{R}}(t_i)|^2 + w^2(\gamma_{\mathbb{R}}(t_{i-1}))\mathrm{d}_X^2(\gamma_X(t_{i-1}), \gamma_X(t_i))},$$

where $\tau := \{0 = t_0, t_1, ..., t_n = 1\}$ is a partition of $I = [0, 1]$ and the limit is taken with respect to the refinement ordering of partitions. It can be proved that the definition above is well posed, i.e. the limit exists. Furthermore, we have the formula

$$l_w[\gamma] = \int_0^1 \sqrt{|\dot{\gamma}_{\mathbb{R}}|^2(t) + w^2(\gamma_{\mathbb{R}}(t))|\dot{\gamma}_X|^2(t)}\,\mathrm{d}t, \tag{1.8}$$

where $|\dot{\gamma}_{\mathbb{R}}|$ and $|\dot{\gamma}_X|$ represent the speed of the curves $\gamma_{\mathbb{R}}, \gamma_X$ respectively.

Then, we can define a pseudo-metric d_w on the space $\mathbb{R} \times X$ as

$$\mathrm{d}_w(A, B) = \inf\{l_w[\gamma] : \gamma \text{ is an absolutely continuous curve from } A \text{ to } B\},$$

where $A, B \in \mathbb{R} \times X$, and we define $\mathbb{R} \times_w X$ as the quotient of $\mathbb{R} \times X$ with respect to the pseudo-metric d_w

Picking w as a constant in the formula (1.8), we can see the above definition for warped products coincides with the definition for product spaces. This observation tells us that we can use the results on cartesian products to study warped products.

Now, we turn to prove our main theorem for warped products, which extends the Theorem-3.18. Our strategy is to 'approximate' a warped product by cartesian products. More precisely, as we can approximate the warping function by piecewise constant functions, we expect to prove that the Sobolev space of the warped product can be equally approximate by the Sobolev space of cartesian products. Using Lemma-3.22, and the formula in Theorem-3.18, we can turn this observation into a rigorous proof.

We then consider the Beppo-Levi space $\mathsf{BL}(X_w)$ defined as follows:

Definition 1.7 (The space $\mathsf{BL}(X_w)$)**.** As a set, $\mathsf{BL}(X_w)$ is the subset of $L^2(X_w, \mathfrak{m}_w)$ made of those functions f such that:

i) for $\mathfrak{m}$-a.e. $x \in X$ we have $f^{(x)} \in W^{1,2}(\mathbb{R}, w_{\mathfrak{m}}\mathcal{L}^1)$,

ii) for $w_{\mathfrak{m}}\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have $f^{(t)} \in W^{1,2}(X)$,

iii) the function

$$|\mathrm{D}f|_w(t,x) := \sqrt{w_{\mathrm{d}}^{-2}(t)|\mathrm{D}f^{(t)}|_X^2(x) + |\mathrm{D}f^{(x)}|_{\mathbb{R}}^2(t)} \tag{1.9}$$

belongs to $L^2(X_w, \mathfrak{m}_w)$.

On $\mathsf{BL}(X_w)$ we put the norm

$$\|f\|_{\mathsf{BL}(X_w)} := \sqrt{\|f\|_{L^2(X_w)}^2 + \||\mathrm{D}f|_w\|_{L^2(X_w)}^2}.$$

We also define the 'local' Beppo-Levi space as

Definition 1.8 (The space $\mathsf{BL}_0(X_w)$)**.** Let $V \subset \mathsf{BL}(X_w)$ be the space of functions f which are identically 0 on $\Omega \times X \subset X_w$ for some open set $\Omega \subset \mathbb{R}$ containing $\{w_{\mathfrak{m}} = 0\}$.

$\mathsf{BL}_0(X_w) \subset \mathsf{BL}(X_w)$ is defined as the closure of V in $\mathsf{BL}(X_w)$.

The goal of Chapter-3 is to compare the spaces $\mathsf{BL}(X_w)$ and $W^{1,2}(X_w)$ and their respective notions of minimal weak upper gradients, namely $|\mathrm{D}f|_w$ and $|\mathrm{D}f|_{X_w}$. Under the sole continuity assumption of $w_{\mathrm{d}}, w_{\mathfrak{m}}$ and the compatibility condition $\{w_{\mathrm{d}} = 0\} \subset \{w_{\mathfrak{m}} = 0\}$ we can prove (see Proposition 3.21 and Proposition 3.24) that

$$\mathsf{BL}_0(X_w) \subset W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$$

and (see Proposition 3.23) that for any $f \in W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$ the identity

$$|\mathrm{D}f|_{X_w} = |\mathrm{D}f|_w$$

holds $\mathfrak{m}_w$-a.e., so that in particular the above inclusions are continuous. Without additional hypotheses it is unclear to us whether $W^{1,2}(X_w) = \mathsf{BL}(X_w)$ (on the other hand, it is easy to construct examples where $\mathsf{BL}_0(X_w)$ is strictly smaller than $\mathsf{BL}(X_w)$). Still, if we assume that

$$\text{the set } \{w_{\mathfrak{m}} = 0\} \subset I \text{ is discrete} \tag{1.10}$$

and that $w_{\mathfrak{m}}$ decays at least linearly near its zeros, i.e.

$$w_{\mathfrak{m}}(t) \leq C \inf_{s: w_{\mathfrak{m}}(s)=0} |t-s|, \qquad \forall t \in \mathbb{R}, \tag{1.11}$$

for some constant $C \in \mathbb{R}$, then we can prove - using capacity arguments - that

$$\mathsf{BL}_0(X_w) = \mathsf{BL}(X_w).$$

Hence that the three spaces considered are all equal. We remark that these two additional assumptions on $w_{\mathfrak{m}}$ are satisfied in all the geometric applications we have in mind, because typically one considers cone/spherical suspensions and in these cases $w_{\mathfrak{m}}$ has at most two zeros and decays polynomially near them.

Then our main theorem is as the following.

Theorem 1.9. *Let $w_{\mathrm{d}}, w_{\mathfrak{m}}$ be warping functions and assume that $w_{\mathfrak{m}}$ has the properties (1.10) and (1.11). Then $W^{1,2}(X_w) = \mathsf{BL}(X_w)$ as sets and for every $f \in W^{1,2}(X_w) = \mathsf{BL}(X_w)$ the identity*

$$|\mathrm{D}f|_{X_w} = |\mathrm{D}f|_w \qquad \mathfrak{m}_w - \ a.e.$$

holds.

From [9] we know that $\mathrm{RCD}(K,\infty)$ spaces have the Sobolev-to-Lipschitz property, i.e. for any function $f \in W^{1,2}$ with $|\mathrm{D}f| \in L^\infty$, there exists a Lipschitz function $\tilde{f}$ such that $f = \tilde{f}$ a.e. and $\mathrm{Lip}(f) = \operatorname{ess\,sup} |\mathrm{D}f|$. It is known in [7, 9] that the cartesian product of RCD spaces is still RCD, therefore it has the Sobolev-to-Lipschitz property.

The Sobolev-to-Lipschitz property builds the connection between Sobolev space and the metric structure. For example, it can be seen that the Sobolev-to-Lipschitz property implies the following duality formula:

$$\mathrm{d}(x,y) = \max\{f(x) - f(y) : |\mathrm{D}f| \le 1\}.$$

This formula is required in the Bakry-Émery theory which we will discuss later. Therefore we would like to know whether we can prove this property for some warped products with less curvature-dimension assumptions.

In Chapter-3, we prove the following theorem:

Theorem 3.30 (Sobolev-to-Lipschitz property) *Let $(X, \mathrm{d}, \mathfrak{m})$ be a doubling $\mathrm{RCD}(K,\infty)$ space, $I \subset \mathbb{R}$ a closed, possibly unbounded interval and $w_{\mathrm{d}}, w_{\mathfrak{m}} : I \to \mathbb{R}$ a pair of warping functions. Assume that $w_{\mathfrak{m}}$ is strictly positive in the interior of I.*

Then the warped product $(X_w, \mathrm{d}_w, \mathfrak{m}_w)$ has the Sobolev to Lipschitz property.

The proof is based on the following property which is satisfied by $\mathrm{RCD}(K,\infty)$ spaces:

Definition 3.28 *We say that $(X, \mathrm{d}, \mathfrak{m})$ is a 'good' space if for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x,y) \in X \times X$ and any $\epsilon > 0$, there exists a family of $\mathcal{W}_2$-absolutely continuous curves $\{\mu_{t,\epsilon}\}_{t\in[0,1]}$ in $\mathcal{P}_2(X)$ with $\mu_{t,\epsilon} < C_\epsilon \mathfrak{m}$ for some positive constant C_ϵ, $\mu_{0,\epsilon} = \frac{1_{B_{r_\epsilon}(x)}}{\mathfrak{m}(B_{r_\epsilon}(x))}\mathfrak{m}$ and $\mu_{1,\epsilon} = \frac{1_{B_{r_\epsilon}(y)}}{\mathfrak{m}(B_{r_\epsilon}(y))}\mathfrak{m}$ such that*

$$\lim_{\epsilon\to 0} l[\{\mu_{t,\epsilon}\}] = \mathrm{d}(x,y), \qquad \lim_{\epsilon\to 0} r_\epsilon = 0.$$

Then, using the ideas and techniques from optimal transport theory, we prove that doubling and 'good' spaces have the Sobolev-to-Lipschitz property. As a consequence, we can prove the Theorem-3.30 above.

1.4 Independence on p of weak upper gradients

In Section-1.2, we introduce a way of constructing the Sobolev space on metric measure spaces. It is known that the p-minimal weak upper gradient of a Sobolev function f, which is denoted by $|\mathrm{D}f|_p$ can be defined equivalently via relaxation of local Lipschitz constants or duality with respect to test plans. Since the map $p \mapsto \|\cdot\|_{L^p}$ is non-decreasing, from the definition we know $p \mapsto |\mathrm{D}f|_p$ is also non-decreasing. This property can also be deduced from the observation that p_1-test plans are always p_2-test plans for $p_1 \geq p_2$.

A natural question about the p-weak gradient is whether $|\mathrm{D}f|_p$ depends on p or not, which means: say that $\mathfrak{m} \in \mathcal{P}(X)$, $1 < p_1 < p_2$, can we say that $|\mathrm{D}f|_{p_1} = |\mathrm{D}f|_{p_2}$ for any $f \in W^{1,p_2}(X, \mathrm{d}, \mathfrak{m})$? Furthermore, if $f \in W^{1,p_1}$ and $f, |\mathrm{D}f|_{p_1} \in L^{p_2}(X)$, can we say that $f \in W^{1,p_2}$?

Let us recall the smooth case. We assume that M is a complete Riemannian manifold, then we can use the Sobolev space using integration by part. The Sobolev space $W^{1,p}$ can be defined in this way: $f \in L^p$ is a Sobolev function if there exists a vector field $g_f \in L^p(TM)$ such that

$$\int (\nabla \cdot v) f \,\mathrm{d}V = \int g_f \cdot v \,\mathrm{d}V \tag{1.12}$$

for any smooth vector fields v.

Another equivalent definition is that there exists a constant C such that

$$\left| \int (\nabla \cdot v) f \,\mathrm{d}V \right| \leq C \|v\|_{L^p(TM)} \tag{1.13}$$

for any smooth vector fields v.

It is known that (1.12) and (1.13) are equivalent when $1 < p < \infty$. In the case of $p = 1$: (1.12) is the definition of $W^{1,1}$ while (1.13) is the definition of the space of BV functions. Moreover, assume $f \in W^{1,p}$ for $p > 1$, we know $f \in W^{1,p'}$ for any $1 \leq p' < p$ and $\|f\|_{W^{1,p}} = (\int |\mathbf{D}f|^p \,\mathrm{d}V)^{\frac{1}{p}}$ where $|\mathbf{D}f|$ is distributional derivative (or the density of the bounded variation of f with respect to the volume measure $\mathrm{d}V$).

In general metric measure spaces, the answers to the above questions are negative (see [36] for a counterexample and also see [6] for a counterexample proposed by Koskela). However, if we assume that the metric measure space is doubling and satisfies a local-Poincaré inequality, the weak gradient is really independent of p as proved by Cheeger in [20]. It has been proved in [34, 35] that $\mathrm{CD}(K,N)$ spaces satisfy these assumptions. In Chapter 4 we prove the identification of p weak gradients for $\mathrm{RCD}(K,\infty)$ spaces, i.e. $\mathrm{CD}(K,\infty)$ spaces which are also infinitesimally Hilbertian. Our result does not depend on finite dimension hypothesis but needs the linearity of the heat flow on $\mathrm{RCD}(K,\infty)$ space. Definition and properties of the heat flow can be found in Chapter-4, Section-4.2.2.

Our idea is to prove the existence of a family of Sobolev functions $\mathcal{A}$ which is dense in energy in $W^{1,2}$, such that for any $f\in\mathcal{A}$, $|\mathrm{D}f|_q=|\mathrm{D}f|_p,\mathfrak{m}$-a.e., then use the lower semicontinuity and monotonicity of $|\mathrm{D}f|_p$ with respect to p to finish the proof. Here, due to the regularity of the heat flow on $\mathrm{RCD}(K,\infty)$ space, we can choose this family $\mathcal{A}$ as the bounded Lipschitz functions.

Let $(X,\mathrm{d},\mathfrak{m})$ be a $\mathrm{RCD}(K,\infty)$ metric measure space. For any bounded Lipchitz function f, we know from [8, 9] that the heat flows $\mathrm{H}_t(f)$ and $\mathrm{H}_t(\mathrm{lip}(f))$ are well defined and $\mathrm{H}_t(f)$ is Lipschitz. By the result in [39] we know

$$\mathrm{lip}(\mathrm{H}_t(f))\le e^{-kt}\mathrm{H}_t(\mathrm{lip}(f)).$$

By definition and lower-semicontinuity we have the following proposition:

Proposition 4.6 *Let* $(X,\mathrm{d},\mathfrak{m})$ *be a* $\mathrm{RCD}(K,\infty)$ *space,* $p\in(1,\infty)$*,* $f\in W^{1,p}(X)$ *and* $t\ge 0$*. Then* $\mathrm{H}_t(f)\in W^{1,p}(X)$ *and*

$$|\mathrm{D}\mathrm{H}_tf|_p^p\le e^{-pKt}\mathrm{H}_t(|\mathrm{D}f|_p^p),\qquad \mathfrak{m}-a.e..$$

Then we can control the local Lipschitz constant of $\mathrm{H}_t(f)$ by its weak gradient.

Proposition 4.7 *Let* $(X,\mathrm{d},\mathfrak{m})$ *be a* $\mathrm{RCD}(K,\infty)$ *space,* $p\in(1,\infty)$*,* $f\in W^{1,p}(X)$ *such that* $f,|\mathrm{D}f|_p\in L^\infty(X)$ *and* $t>0$*. Then* $\mathrm{H}_t(f)$ *is Lipschitz and*

$$\mathrm{lip}(\mathrm{H}_t(f))\le e^{-Kt}\sqrt[p]{\mathrm{H}_t(|\mathrm{D}f|_p^p)},\qquad \textit{pointwise on } X.$$

As we know that the local Lipschitz constant is no less than the weak gradient, we can prove the identification result for Lipschitz functions. This proposition extends the result proved in [9] for 2-weak gradients.

Proposition 4.8 *Let $p, q \in (1, \infty)$ and $f \in \mathrm{Lip}\, X$. Then*

$$|\mathrm{D}f|_q = |\mathrm{D}f|_p, \qquad \mathfrak{m} - \ a.e..$$

Finally, we can use the monotonicity for weak gradients, i.e. $|\mathrm{D}f|_{p_1} \leq |\mathrm{D}f|_{p_2}$ for any $1 < p_1 < p_2$, to prove the main theorem by a relaxation procedure and lower-semicontinuity:

Theorem 4.9 (Identification of weak upper gradients) *Let $p, q \in (1, \infty)$ and $f \in S^p_{\mathrm{loc}}(X)$ such that $|\mathrm{D}f|_p \in L^q_{\mathrm{loc}}(X)$. Then $f \in S^q_{\mathrm{loc}}(X)$ and*

$$|\mathrm{D}f|_q = |\mathrm{D}f|_p, \qquad \mathfrak{m} - \ a.e...$$

1.5 Bakry-Émery's theory and Ricci tensor

Let M be a Riemannian manifold equipped with a metric tensor $\langle \cdot, \cdot \rangle : [TM]^2 \mapsto C^\infty(M)$. We have the Bochner formula

$$\Gamma_2(f) = \mathrm{Ricci}(\nabla f, \nabla f) + \|\mathrm{H}_f\|^2_{\mathrm{HS}}, \tag{1.14}$$

valid for any smooth function f, where $\|\mathrm{H}_f\|_{\mathrm{HS}}$ is the Hilbert-Schimidt norm of the Hessian $\mathrm{H}_f := \nabla \mathrm{d} f$ and the operator Γ_2 is defined by

$$\Gamma_2(f) := \frac{1}{2} L\Gamma(f, f) - \Gamma(f, Lf), \qquad \Gamma(f, f) := \frac{1}{2} L(f^2) - fLf$$

where $\Gamma(\cdot, \cdot) = \langle \nabla \cdot, \nabla \cdot \rangle$, and $L = \Delta$ is the Laplace-Beltrami operator.

In particular, if the Ricci curvature of M is bounded from below by K, i.e. $\mathrm{Ricci}(v, v)(x) \geq K|v|^2(x)$ for any $x \in M$ and $v \in T_x M$, and the dimension is bounded from above by $N \in [1, \infty]$, we have the Bochner inequality

$$\Gamma_2(f) \geq \frac{1}{N}(\Delta f)^2 + K\Gamma(f). \tag{1.15}$$

Conversely, it is not hard to show that the validity of (1.15) for any smooth function f implies that the manifold has lower Ricci curvature bound K and upper dimension bound N, or in short that it is a $\mathrm{CD}(K,N)$ manifold.

Being this characterization of the $\mathrm{CD}(K,N)$ condition only based on properties of L, one can take (1.15) as definition of what it means for a diffusion operator L to satisfy the $\mathrm{CD}(K,N)$ condition. This was the approach suggested by Bakry-Émery in [15], we refer to [16] for an overview on the subject.

Following this line of thought, one can wonder whether in this framework one can recover the definition of the Ricci curvature tensor and deduce from (1.15) that it is bounded from below by K. From (1.14) we see that a natural definition is

$$\mathrm{Ricci}(\nabla f, \nabla f) := \Gamma_2(f) - \|\mathrm{H}_f\|^2_{\mathrm{HS}}, \tag{1.16}$$

and it is clear that if $\mathrm{Ricci} \geq K$, then (1.15) holds with $N = \infty$. There are few things that need to be understood in order to make definition (1.16) rigorous and complete in the setting of diffusion operators:

1) If our only data is the diffusion operator L, how can we give a meaning to the Hessian term in (1.16)?
2) Can we deduce that the Ricci curvature defined as in (1.16) is actually bounded from below by K from the assumption (1.15)?
3) Can we include the upper bound on the dimension in the discussion? How the presence of N affects the definition of the Ricci curvature?

This last question has a well known answer: it turns out that the correct thing to do is to define, for every $N \geq 1$, a sort of 'N-dimensional' Ricci tensor as follows:

$$\mathrm{Ricci}_N(\nabla f, \nabla f) := \begin{cases} \Gamma_2(f) - \|\mathrm{H}_f\|^2_{\mathrm{HS}} - \frac{1}{N-n(x)}(\mathrm{tr}\mathrm{H}_f - Lf)^2, & \text{if } N > n, \\ \Gamma_2(f) - \|\mathrm{H}_f\|^2_{\mathrm{HS}} - \infty(\mathrm{tr}\mathrm{H}_f - Lf)^2, & \text{if } N = n, \\ -\infty, & \text{if } N < n, \end{cases} \tag{1.17}$$

where n is the dimension of the manifold (recall that on a weighted manifold in general we have $\mathrm{tr}\mathrm{H}_f \neq \Delta f$). It is then not hard to see that if $\mathrm{Ricci}_N \geq K$ then indeed (1.15) holds.

It is harder to understand how to go back and prove that $\mathrm{Ricci}_N \geq K$ starting from (1.15). A first step in this direction, which answers (1), is to notice that in the smooth setting the identity

$$2\mathrm{H}_f(\nabla g, \nabla h) = \Gamma(g, \Gamma(f, h)) + \Gamma(h, \Gamma(f, g)) - \Gamma(f, \Gamma(g, h))$$

for any smooth g, h characterizes the Hessian of f, so that the same identity can be used to define the Hessian starting from a diffusion operator only. The question is then whether one can prove any efficient bound on it starting from (1.15) only. The first results in this direction where obtained by Bakry in [13] and [14], and only recently Sturm [42] concluded the argument showing that (1.15) implies $\mathrm{Ricci}_N \geq K$. In Sturm's approach, the operator Ricci_N is not defined as in (1.17), but rather as

$$\mathrm{Ricci}_N(\nabla f, \nabla f)(x) := \inf_{g\,:\,\Gamma(f-g)(x)=0} \Gamma_2(g)(x) - \frac{(Lg)^2(x)}{N} \tag{1.18}$$

and it is part of his contribution the proof that this definition is equivalent to (1.17).

All this for smooth, albeit possibly abstract, structures. On the other hand, there is as of now a quite well established theory of (non-smooth) metric measure spaces satisfying a curvature-dimension condition: that of $\mathrm{RCD}^*(K, N)$ spaces introduced by Ambrosio-Gigli-Savaré (see [9] and [24]) as a refinement of the original intuitions of Lott-Sturm-Villani ([35] and [40, 41]) and Bacher-Sturm ([12]). In this setting, there is a very natural Laplacian and inequality (1.15) is known to be valid in the appropriate weak sense (see [9] and [21]) and one can therefore wonder if even in this low-regularity situation one can produce an effective notion of N-Ricci curvature. Part of the problem here is the a priori lack of vocabulary, so that for instance it is unclear what a vector field should be.

In the recent paper [23], Gigli builds a differential structure on metric measure spaces suitable to handle the objects we are discussing (see the preliminary section of Chapter-5 for some details). One of his results is to give a meaning to formula (1.16) on $\mathrm{RCD}(K, \infty)$ spaces and to prove that the resulting Ricci curvature tensor, now measure-valued, is bounded from below by K. Although giving comparable results, we remark that the definitions used in [23] are different from those in [42]: it is indeed unclear how to give a meaning to formula (1.18) in the non-smooth setting, so that in [23] the definition (1.16) has been adopted.

Gigli worked solely in the $\mathrm{RCD}(K, \infty)$ setting. The contribution of the current work is to adapt Gigli's tool and Sturm's computations to give a complete description of the N-Ricci curvature tensor on $\mathrm{RCD}^*(K, N)$ spaces for $N < \infty$.

Our main result is the fact that the N-Ricci curvature is bounded from below by K on a $\mathrm{RCD}(K', \infty)$ space if and only if the space is $\mathrm{RCD}^*(K, N)$.

Now we introduce some notations and necessary backgrounds of non-smooth Bakry-Émery theory. More details can be found in [7] (for $\mathrm{RCD}(K,\infty)$ space) and [21] (for $\mathrm{RCD}^*(K,N)$ space) where the authors construct the non-smooth counterparts of the Bakry-Émery theory, which builds the link between the Bakry-Émery theory and Lott-Sturm-Villani's theory based on optimal transport.

We shall denote by $\langle \nabla f, \nabla g\rangle$ the carré du champ associated to the canonical Dirichlet form (or call it Cheeger energy) on a $\mathrm{RCD}(K,\infty)$ space $(X,\mathrm{d},\mathfrak{m})$. We then define the space $\mathrm{D}(\mathbf{\Delta}) \subset W^{1,2}(X)$ as the space of $f \in W^{1,2}(X)$ such that there exists a measure μ satisfying

$$\int h\,\mu = -\int \langle \nabla h, \nabla f\rangle\,\mathfrak{m}, \forall h : X \mapsto \mathbb{R},\ \text{Lipschitz with bounded support.}$$

In this case the measure μ is unique and we shall denote it by $\mathbf{\Delta} f$. If $\mathbf{\Delta} f \ll m$, we still denote its density by Δf.

We define the set $\mathrm{TestF}(X) \subset W^{1,2}(X)$ of test functions as

$$\mathrm{TestF}(X) := \{f \in \mathrm{D}(\Delta) \cap L^\infty : |\mathrm{D}f| \in L^\infty \quad \textit{and} \quad \Delta f \in W^{1,2}(X)\}.$$

It is proved in [9], [23] that $\mathrm{TestF}(X)$ is dense in $W^{1,2}(X)$.

For $f \in \mathrm{TestF}(X)$ we define the measure $\mathbf{\Gamma}_2(f)$

$$\mathbf{\Gamma}_2(f) = \frac{1}{2}\mathbf{\Delta}|\mathrm{D}f|^2 - \langle f, \Delta f\rangle\,\mathfrak{m},$$

and we define the Hessian of $f \in \mathrm{TestF}(X)$ as

$$\mathrm{H}_f(\nabla g, \nabla h) = \frac{1}{2}\big(\langle \nabla\langle \nabla f, \nabla g\rangle, \nabla h\rangle + \langle \nabla\langle \nabla f, \nabla h\rangle, \nabla g\rangle - \langle \nabla\langle \nabla g, \nabla h\rangle, \nabla f\rangle\big),$$

for every $g, h \in \mathrm{TestF}(X)$.

The interest of this vocabulary is in the fact that, like in the case of smooth manifolds, it can be used in the non-smooth context to characterize lower Ricci bounds. Indeed, as proved in [7] and [21], on a $\mathrm{RCD}^*(K,N)$ space, the following Bochner type inequality (1.19) holds for every $f \in \mathrm{TestF}(X)$.

$$\mathbf{\Gamma}_2(f) \geq \Big(K|\mathrm{D}f|^2 + \frac{1}{N}(\Delta f)^2\Big)\,\mathfrak{m}. \tag{1.19}$$

Conversely, if an infinitesimally Hilbertian space $(X,\mathrm{d},\mathfrak{m})$ satisfies the Sobolev-to-Lipschitz property and the above inequality holds in the appropriate weak sense for sufficient many f's, then it is a $\mathrm{RCD}^*(K,N)$ space.

Inequality (1.19) has been improved in [23] in the case $N = \infty$ to incorporate the Hessian of the functions:

$$\mathbf{\Gamma}_2(f) \geq \left(K|\mathrm{D}f|^2 + \|\mathrm{H}_f\|_{\mathrm{HS}}^2\right)\mathfrak{m}$$

for every $f \in \mathrm{TestF}(X)$. See [23] for the definition of the Hilbert-Schmidt norm of the Hessian.

In [42], under additional smoothness assumptions, the analysis has been pushed further to incorporate informations coming from the finite dimensionality. It is then natural to ask whether these results can be proved in the full generality of $\mathrm{RCD}^*(K,N)$ spaces. The difficulty of extending the result in [42] is the lack smooth tangent fields and smooth tensors on a non-smooth metric measure space. In [23], Gigli defines the L^∞-module $L^2(TM)$ as the non-smooth counterpart of the tangent bundle and defines the non-smooth tensors in the viewpoint of L^∞-module. In Chapter-5 we use these tools to study the differential structure of $\mathrm{RCD}^*(K,N)$ space. More details can be found in the preliminary part of Chapter-5 (or see [23] for the relevant definitions).

First of all, we study the dimension of a $\mathrm{RCD}^*(K,N)$ space which is understood as the dimension of $L^2(TM)$ as a $L^\infty(M)$-module.

Proposition 5.12 *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}^*(K,N)$ metric measure space, then $\mathrm{dim}M \leq N$. Furthermore, if the local dimension $\mathrm{dim}_{\mathrm{loc}}$ on a Borel set E is N, then $\mathrm{trH}_f(x) = \Delta f(x)$ $\mathfrak{m}$-a.e. $x \in E$ for $f \in \mathrm{TestF}$.*

Notice that the equality $\mathrm{trH}_f = \Delta f$ does not hold even in smooth metric measure spaces (for example, weighted Riemannian manifolds). In this case, the term $\frac{1}{N-\mathrm{dim}_{\mathrm{loc}}}(\mathrm{trH}_f - \Delta f)^2$ is not trivial and makes sense on a $\mathrm{RCD}^*(K,N)$ space.

Then by a variational argument and change of variables formula in [39], we prove the following theorem:

Theorem 5.13 *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}^*(K,N)$ metric measure space. Then*

$$\mathbf{\Gamma}_2(f) \quad \geq \quad \left(K|\mathrm{D}f|^2 + \|\mathrm{H}_f\|_{\mathrm{HS}}^2 + \frac{1}{N - \mathrm{dim}_{\mathrm{loc}}}(\mathrm{trH}_f - \Delta f)^2\right)\mathfrak{m}$$

holds for any $f \in \mathrm{TestF}$, where $\frac{1}{N-\mathrm{dim}_{\mathrm{loc}}}(\mathrm{trH}_f - \Delta f)^2)$ is taken 0 by definition on the set $\{x : \mathrm{dim}_{\mathrm{loc}}(x) = N\}$.

As an application of the theorem above, we can define the Ricci tensor $\mathbf{Ricci}_N(\nabla f, \nabla f)$ as:

$$\mathbf{Ricci}_N(\nabla f, \nabla f) := \mathbf{\Gamma}_2(f) - \|\mathrm{H}_f\|_{\mathrm{HS}}^2\,\mathfrak{m} - \mathbf{R}_N(f)$$

where

$$\mathbf{R}_N(f) := \begin{cases} \frac{1}{N-n(x)}\big(\mathrm{trH}_f - \Delta f)^2\big)\,\mathfrak{m}, & \text{if } n < N, \\ +\infty(\mathrm{trH}_f - \Delta f)^2\,\mathfrak{m} & \text{if } n = N, \\ +\infty, & \text{if } n > N. \end{cases}$$

It can be seen easily that the definition above fulfills all the requirement for $\mathbf{Ricci}_N$. In particular, we can rewrite the inequality in Theorem 5.13 as:

$$\mathbf{\Gamma}_2(f) \geq \mathbf{Ricci}_N(\nabla f, \nabla f) + \big(\|\mathrm{H}_f\|^2_{\mathrm{HS}} + \frac{1}{N - \dim_{\mathrm{loc}}}(\mathrm{trH}_f - \Delta f)^2\big)\,\mathfrak{m}.$$

Furthermore, using the vocabulary in [23] we can extend the results above and the definition of the N-Ricci tensor to more general tangent fields, more details about $\mathrm{TestV}(M), H^{1,2}_H(TM)$ can be found in Chapter-5 or [23].

Definition 5.15 (Ricci tensor) *We define* $\mathbf{Ricci}_N$ *as a measure valued continuous map on* $[H^{1,2}_H(TM)]^2$ *such that for any* $X, Y \in \mathrm{TestV}(M)$ *it holds*

$$\mathbf{Ricci}_N(X,Y) \quad = \quad \mathbf{\Gamma}_2(X,Y) - \langle (\nabla X)^b, (\nabla Y)^b \rangle_{\mathrm{HS}}\,\mathfrak{m} - R_N(X,Y)\,\mathfrak{m}.$$

where

$$\mathbf{\Gamma}_2(X,Y) := \mathbf{\Delta}\frac{\langle X, Y\rangle}{2} + \Big(\frac{1}{2}\langle X, (\Delta_{\mathrm{H}} Y^b)^\sharp\rangle + \frac{1}{2}\langle Y, (\Delta_{\mathrm{H}} X^b)^\sharp\rangle\Big)\,\mathfrak{m},$$

and

$$R_N(X,Y) := \begin{cases} \frac{1}{N-\dim_{\mathrm{loc}}}\big(\mathrm{tr}(\nabla X)^b - \mathrm{div} X\big)\big(\mathrm{tr}(\nabla Y)^b - \mathrm{div} Y\big) & \dim_{\mathrm{loc}} < N, \\ 0 & \dim_{\mathrm{loc}} \geq N. \end{cases}$$

It can be seen that $\mathbf{Ricci}_N$ is a well defined tensor, i.e. $(X,Y) \mapsto \mathbf{Ricci}_N(X,Y)$ is a symmetric $\mathrm{TestF}(M)$-bilinear form. Then we prove the following theorem.

Theorem 5.16 *Let* M *be a* $\mathrm{RCD}^*(K,N)$ *space. Then*

$$\mathbf{Ricci}_N(X,X) \geq K|X|^2\,\mathfrak{m},$$

and

$$\mathbf{\Gamma}_2(X,X) \quad \geq \quad \Big(\frac{(\mathrm{div} X)^2}{N} + \mathbf{Ricci}_N(X,X)\Big)\,\mathfrak{m}$$

holds for any $X \in H^{1,2}_H(TM)$. *Conversely, on a* $\mathrm{RCD}(K',\infty)$ *space* M, *assume that*

(1) $\dim M \leq N$

(2) $\mathrm{tr}(\nabla X)^b = \mathrm{div} X \ \mathfrak{m} - a.e. \ on \ \{\dim_{\mathrm{loc}} = N\}, \forall X \in H^{1,2}_H(TM)$

(3) $\mathbf{Ricci}_N \geq K$

for some $K \in \mathbb{R}$, $N \in [1, +\infty]$, *then it is* $\mathrm{RCD}^*(K, N)$.

Chapter 2

The continuity equation on metric measure spaces

Abstract

In this chapter, we show that it makes sense to write the continuity equation on a metric measure space $(X, \mathrm{d}, \mathfrak{m})$, and that absolutely continuous curves $\{\mu_t\}_t$ w.r.t. the distance $\mathcal{W}_2$ can be completely characterized as solutions of the continuity equation itself, provided we impose the condition $\mu_t \leq C\mathfrak{m}$ for every t and some $C > 0$. We also show that our frameworks are adaptable to several classical results.

Résumé

Dans ce chapitre, nous montrons qu'il est possible d'écrire l'équation de continuité dans un espace métrique mesuré $(X, \mathrm{d}, \mathfrak{m})$, et que les courbes absolument continues $\{\mu_t\}_{t\in[0,1]}$ par rapport à la distance $\mathcal{W}_2$ peut être complètement caractérisé comme solutions de l'équation de continuité lui-même, sous la condition $\mu_t \leq C\mathfrak{m}$ pour chaque $t \in [0, 1]$ et certains $C > 0$. Nous montrons également que nos cadres sont adaptables à plusieurs résultats classiques.

The results in this chapter are contained in [25].

2.1 Introduction

A crucial intuition of Otto [37], inspired by the work of Benamou-Brenier [17], has been to realize that absolutely continuous curves of measures $\{\mu_t\}_t$ w.r.t. the quadratic transportation distance $\mathcal{W}_2$ on $\mathbb{R}^d$ can be interpreted as solutions of the continuity equation

$$\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0, \tag{2.1}$$

where the vector fields v_t should be considered as the 'velocity' of the moving mass μ_t and, for curves with square-integrable speed, satisfy

$$\int_0^1 \int |v_t|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t < \infty. \tag{2.2}$$

This intuition has been made rigorous by the first author, Ambrosio and Savaré in [5], where it has been used to develop a solid first order calculus on the space $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$, with particular focus on the study of gradient flows.

Heuristically speaking, the continuity equation describes the link existing between the 'vertical derivative' $\partial_t \mu_t$ (think to it as variation of the densities, for instance) and the 'horizontal displacement' v_t. In this sense it provides the crucial link between analysis made on the L^p spaces, where the distance is measured 'vertically', and the one based on optimal transportation, where distances are measured by 'horizontal' displacement. This is indeed the heart of the crucial substitution made by Otto in [37] who, to define the metric tensor g_μ on the space $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$ at a measure $\mu = \rho \mathcal{L}^d$ considers a 'vertical' variation $\delta\rho$ such that $\int \delta\rho \, \mathrm{d}\mathcal{L}^d = 0$, then looks for solutions of

$$\delta\rho = -\nabla \cdot (\nabla\varphi\rho), \tag{2.3}$$

and finally defines

$$g_\mu(\delta\rho, \delta\rho) := \int |\nabla\varphi|^2 \, \mathrm{d}\mu. \tag{2.4}$$

The substitution (2.3) is then another way of thinking at the continuity equation, while the definition (2.4) corresponds to the integrability requirement (2.2).

It is therefore not surprising that each time one wants to put in relation the geometry of optimal transport with that of L^p spaces some form of continuity equation must be studied. In the context of analysis on non-smooth structures, this has been implicitly done in [9, 28] to show that the gradient flow of the relative entropy on the space $(\mathcal{P}_2(X), \mathcal{W}_2)$ produces the same evolution of the gradient flow of the energy (sometime called Cheeger energy or Dirichlet energy) in the space $L^2(X, \mathfrak{m})$, where $(X, \mathrm{d}, \mathfrak{m})$ is some given metric measure space.

The purpose of this paper is to make these arguments more explicit and to show that:

i) It is possible to formulate the continuity equation on general metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$,

ii) Solutions of the continuity equation completely characterize absolutely continuous curves $\{\mu_t\}_t \subset \mathcal{P}(X)$ with square-integrable speed w.r.t. $\mathcal{W}_2$ and such that $\mu_t \leq C\mathfrak{m}$ for every $t \in [0,1]$ and some $C > 0$.

In fact, the techniques we use can directly produce similar results for the distances $\mathcal{W}_p$, $p \in (1,\infty)$, and for curves whose speed is in L^1 rather then in some L^p, $p > 1$. Yet, we prefer not to discuss the full generality in order to concentrate on the main ideas.

Let us discuss how to formulate the continuity equation on a metric measure space where no a priori smooth structure is available. Notice that in the smooth setting (2.1) has to be understood in the sense of distributions. If we assume weak continuity of $\{\mu_t\}_t$, this is equivalently formulated as the fact that that for every $f \in C_c^\infty(\mathbb{R}^d)$ the map $t \mapsto \int f\,\mathrm{d}\mu_t$ is absolutely continuous and the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f\,\mathrm{d}\mu_t = \int \mathrm{d}f(v_t)\,\mathrm{d}\mu_t,$$

holds for a.e. $t \in [0,1]$. In other words, the vector fields v_t only act on differential of smooth functions and can therefore be thought of as linear functionals L_t from the space of differentials of smooth functions to $\mathbb{R}$. Recalling (2.2), the norm $\|L_t\|^*_{\mu_t}$ of L_t should be defined as

$$\frac{1}{2}(\|L_t\|^*_{\mu_t})^2 = \sup_{f \in C_c^\infty(\mathbb{R}^d)} L_t(f) - \frac{1}{2}\int |\mathrm{d}f|^2\,\mathrm{d}\mu_t,$$

so that being $\{\mu_t\}_t$ 2-absolutely continuous is equivalent to require that $t \mapsto \|L_t\|^*_{\mu_t} \in L^2(0,1)$.

Seeing the continuity equation in this way allows for a formulation of it in the abstract context of metric measure spaces $(X,\mathrm{d},\mathfrak{m})$. Indeed, recall that there is a well established notion of 'space of functions having distributional differential in $L^2(X,\mathfrak{m})$', which we will denote by $S^2(X) = S^2(X,\mathrm{d},\mathfrak{m})$ and that for each function $f \in S^2(X)$ it is well defined the 'modulus of the distributional differential $|\mathrm{D}f| \in L^2(X,\mathfrak{m})$'.

Then given a linear map $L : S^2(X) \mapsto \mathbb{R}$ and μ such that $\mu \leq C\mathfrak{m}$ for some $C > 0$ we can define the norm $\|L\|^*_\mu$ as

$$\frac{1}{2}(\|L\|^*_\mu)^2 := \sup_{f \in S^2(X)} L(f) - \frac{1}{2}\int |\mathrm{D}f|^2\,\mathrm{d}\mu. \tag{2.5}$$

Hence given a curve $\{\mu_t\}_t \subset \mathcal{P}(X)$ such that $\mu_t \leq C\mathfrak{m}$ for some $C > 0$ and every $t \in [0,1]$ and a family $\{L_t\}_{t\in[0,1]}$ of maps from $S^2(X)$ to $\mathbb{R}$ such that $\int_0^1 (\|L_t\|^*_{\mu_t})^2\,\mathrm{d}t < \infty$, we can say that the curve $\{\mu_t\}_t \subset \mathcal{P}(X)$ solves the continuity equation

$$\partial_t \mu_t = L_t,$$

provided:

i) for every $f \in S^2(X)$ the map $t \mapsto \int f \,\mathrm{d}\mu_t$ is absolutely continuous,

ii) the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f \,\mathrm{d}\mu_t = L_t(f), \tag{2.6}$$

holds for a.e. t.

Then we show that such formulation of the continuity equation fully characterizes absolutely continuous curves $\{\mu_t\}_t$ with square-integrable speed on the space $(\mathcal{P}_2(X), \mathcal{W}_2)$, provided we restrict the attention to curves such that $\mu_t \leq C\mathfrak{m}$ for some $C > 0$ and every $t \in [0, 1]$. See Theorem 4.9.

Concerning the proof of this result, we remark that the implication from absolute continuity of $\{\mu_t\}_t$ to the 'PDE' (2.6) is quite easy to establish and follows essentially from the definition of Sobolev functions. This is the easy implication even in the smooth context whose proof carries over quite smoothly to the abstract setting, the major technical difference being that we don't know if in general the space $S^2(X)$ is separable or not, a fact which causes some complications in the way we can really write down the equation (2.6), see Definition 2.10.

The converse one is more difficult, as it amounts in proving that the differential identity (2.6) is strong enough to guarantee absolute continuity of the curve. The method used in the Euclidean context consists in regularizing the curve, apply the Cauchy-Lipschitz theory to the approximating sequence to find a flow of the approximating vector fields which can be used to transport μ_t to μ_s and finally in passing to the limit. By nature, this approach cannot be used in non-smooth situations. Instead, we use a crucial idea due to Kuwada which has already been applied to study the heat flow [8, 28]. It amounts in passing to the dual formulation of the optimal transport problem by noticing that

$$\frac{1}{2}\mathcal{W}_2^2(\mu_1, \mu_0) = \sup \int Q_1\varphi \,\mathrm{d}\mu_1 - \int \varphi \,\mathrm{d}\mu_0, \tag{2.7}$$

the sup being taken among all Lipschitz and bounded $\varphi : X \mapsto \mathbb{R}$, where $Q_t\varphi$ is the evolution of φ via the Hopf-Lax formula. A general result obtained in [8] has been that it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_t\varphi(x) + \frac{\mathrm{lip}(Q_t\varphi)^2(x)}{2} \leq 0, \tag{2.8}$$

for every t except a countable number, where $\mathrm{lip}(f)$ is the local Lipschitz constant f. Thus we can formally write

$$\begin{aligned}
\int Q_1\varphi \,\mathrm{d}\mu_1 - \int \varphi \,\mathrm{d}\mu_0 &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\int Q_t\varphi \,\mathrm{d}\mu_t \,\mathrm{d}t \\
\text{by (2.6)} \qquad &= \int_0^1 \int \frac{\mathrm{d}}{\mathrm{d}t} Q_t\varphi \,\mathrm{d}\mu_t \,\mathrm{d}t + \int_0^1 L_t(Q_t\varphi)\,\mathrm{d}t, \\
\text{by (2.5), (2.8)} \qquad &\leq \int_0^1 -\frac{\mathrm{lip}(Q_t\varphi)^2}{2}\,\mathrm{d}\mu_t \,\mathrm{d}t + \frac{1}{2}\int_0^1 (\|L_t\|^*_{\mu_t})^2 \,\mathrm{d}t + \frac{1}{2}\int_0^1 \int |\mathrm{D}Q_t\varphi|^2 \,\mathrm{d}\mu_t \,\mathrm{d}t.
\end{aligned}$$

Using the fact that $|\mathrm{D}f| \leq \mathrm{lip}(f)$ $\mathfrak{m}$-a.e. for every Lipschitz f we then conclude that

$$\int Q_1\varphi \,\mathrm{d}\mu_1 - \int \varphi \,\mathrm{d}\mu_0 \leq \frac{1}{2}\int_0^1 (\|L_t\|^*_{\mu_t})^2 \,\mathrm{d}t.$$

Here the right hand side does not depend on φ, hence by (2.7) we deduce

$$\mathcal{W}_2^2(\mu_1, \mu_0) \leq \int_0^1 (\|L_t\|^*_{\mu_t})^2 \,\mathrm{d}t,$$

which bounds $\mathcal{W}_2$ in terms of the L_t's only. Replacing $0, 1$ with general $t, s \in [0, 1]$ we deduce the desired absolute continuity. As presented here, the computation is only formal, but a rigorous justification can be given, thus leading to the result. See the proof of Theorem 4.9.

It is worth pointing out that Kuwada's lemma works even if we don't know any uniqueness result for the initial value problem (2.6). That is we don't know if given μ_0 and a family of operators L_t from $S^2(X)$ to $\mathbb{R}$ the solution of (2.6) is unique or not, because we "can't follow the flow of the L_t's". Yet, it is possible to deduce anyway that any solution is absolutely continuous.

It is also worth to make some comments about the assumption $\mu_t \leq C\mathfrak{m}$. Notice that if we don't impose any condition on the μ_t's, we could consider curves of the kind $t \mapsto \delta_{\gamma_t}$, where γ is a given Lipschitz curve. In the smooth setting we see that such curve solves

$$\partial_t \delta_{\gamma_t} + \nabla \cdot (\gamma_t' \delta_{\gamma_t}) = 0,$$

so that to write the continuity equation for such curve amounts to know the value of γ_t' at least for a.e. t. In the non-smooth setting to do this would mean to know who is the tangent space at γ_t for a.e. t along a Lipschitz curve γ, an information which without any assumption on X seems quite too strong. Instead, the process of considering only measures with bounded density has the effect of somehow 'averaging out the unsmoothness of the space' and allows for the possibility of building a working differential calculus, a point raised and heavily used in [22]. Here as application of the continuity equation to differential calculus we provide a Benamou-Brenier formula and describe the derivative of $\frac{1}{2}W_2^2(\cdot,\nu)$ along an absolutely continuous curve.

We then study situations where the operators L_t can be given somehow more explicitly. Recall that on the Euclidean setting the optimal (in the sense of energy-minimizer) vector fields v_t appearing in (2.1) always belong to the $L^2(\mu_t)$-closure of the set of gradients of smooth functions and that is some case they are really gradient of functions, so that (2.1) can be written as

$$\partial_t\mu_t + \nabla\cdot(\nabla\phi_t\mu_t) = 0, \tag{2.9}$$

for some given smooth $\{\phi_t\}_{t\in[0,1]}$, which means that for f smooth it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f\,\mathrm{d}\mu_t = \int \mathrm{d}f(\nabla\phi_t)\,\mathrm{d}\mu_t.$$

To interpret the equation (2.9) in the abstract framework we need to understand the duality relation between differentials and gradients of Sobolev functions on metric measure spaces. This has been done in [24], where for given $f,g\in S^2(X)$ the two functions $\mathrm{D}^-f(\nabla g)$ and $\mathrm{D}^+f(\nabla g)$ have been introduced. If the space is a Riemannian manifold or a Finsler one with norms strictly convex, then we have $\mathrm{D}^-f(\nabla g) = \mathrm{D}^+(\nabla g)$ a.e. for every f,g, these being equal to the value of the differential of f applied to the gradient of g obtained by standard means. In the general case we do not have such single-valued duality, due to the fact that even in a flat normed situation the gradient of a function is not uniquely defined should the norm be not strictly convex. Thus the best we can do is to define $\mathrm{D}^-f(\nabla g)$ and $\mathrm{D}^+f(\nabla g)$ as being, in a sense, the minimal and maximal value of the differential of f applied to all the gradients of g.

Then we can interpret (2.9) in the non-smooth situation by requiring that for $f\in S^2(X)$ it holds

$$\int \mathrm{D}^-f(\nabla\phi_t)\,\mathrm{d}\mu_t \le \frac{\mathrm{d}}{\mathrm{d}t}\int f\,\mathrm{d}\mu_t \le \int \mathrm{D}^+f(\nabla\phi_t)\,\mathrm{d}\mu_t, \qquad a.e.\ t,$$

and it turns out that this way of writing the continuity equation, which requires two inequalities rather than an equality, is still sufficient to grant absolute continuity of the curve.

Notice that in the Euclidean setting, if the functions ϕ_t are smooth enough we can construct the flow associated to $\nabla\phi_t$ by solving

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}T(x,t,s) &= \nabla\phi_t(T(x,t,s)), \\[2ex] T(x,t,t) &= x, \end{cases}$$

so that the curves $t \mapsto T(x,t,s)$ are gradient flows of the evolving function ϕ_t and a curve $\{\mu_t\}_t$ solves (2.6) if and only if $\mu_t = T(\cdot,t,0)_\sharp\mu_0$ for every $t \in [0,1]$. Interestingly enough, this point of view can be made rigorous even in the setting of metric measure spaces and a similar characterization of solutions of (2.6) can be given, see Theorem 2.23.

We conclude the paper by showing that the heat flows and the geodesics satisfy the same sort of continuity equation the satisfy in the smooth case, namely

$$\partial_t\mu_t + \nabla\cdot(\nabla(-\log(\rho_t))\mu_t) = 0,$$

for the heat flow, where $\mu_t = \rho_t\mathfrak{m}$, and

$$\partial_t\mu_t + \nabla\cdot(\nabla\phi_t\mu_t) = 0,$$

with $\phi_t = -Q_{1-t}(-\varphi^c)$ for the geodesics, where φ is a Kantorovich potential inducing the geodesic. Here the aim is not to prove new results, as these two examples were already considered in the literature [8, 9, 22, 28], but rather to show that they are compatible with the theory we propose. We also discuss in which sense and under which circumstances an heat flow and a geodesic can be considered not just as absolutely continuous curves on $(\mathcal{P}_2(X), \mathcal{W}_2)$, but rather as C^1 curves.

2.2 Preliminaries

2.2.1 Metric spaces and optimal transport

We quickly recall here those basic fact about analysis in metric spaces and optimal transport we are going to use in the following. Standard references are [5], [43] and [3].

Let (X,d) be a metric space. Given $f : X \mapsto \mathbb{R}$ the local Lipschitz constant $\mathrm{lip}(f) : X \mapsto [0,\infty]$ is defined as

$$\mathrm{lip}(f)(x) := \varlimsup_{y\to x}\frac{|f(y)-f(x)|}{\mathrm{d}(x,y)},$$

if x is not isolated and 0 otherwise. Recall that the Lipschitz constant of Lipschitz function is defined as:

$$\mathrm{Lip}(f) := \sup_{x\neq y}\frac{|f(y)-f(x)|}{\mathrm{d}(x,y)}$$

In particular, if (X,d) is a geodesic space, we have $\mathrm{Lip}(f) = \sup_x \mathrm{lip}(f)(x)$.

A curve $\gamma : [0,1] \mapsto X$ is said absolutely continuous provided there exists $f \in L^1(0,1)$ such that

$$\mathrm{d}(\gamma_s, \gamma_t) \leq \int_t^s f(r)\,\mathrm{d}r, \qquad \forall t, s \in [0,1],\ t < s. \tag{2.10}$$

For an absolutely continuous curve γ it can be proved that the limit

$$\lim_{h \to 0} \frac{\mathrm{d}(\gamma_{t+h}, \gamma_t)}{|h|},$$

exists for a.e. t and thus defines a function, called metric speed and denoted by $|\dot{\gamma}_t|$, which is in $L^1(0,1)$ and is minimal, in the a.e. sense, among the class of L^1-functions f for which (2.10) holds.

If there exists $f \in L^2(0,1)$ for which (2.10) holds, we say that the curve is 2-absolutely continuous (2-a.c. in short). In the following we will often write $\int_0^1 |\dot{\gamma}_t|^2\,\mathrm{d}t$ for a curve γ which a priori is only continuous: in this case the value of the integral is taken by definition $+\infty$ if γ is not absolutely continuous.

The space of continuous curves on $[0,1]$ with values in X will be denoted by $C([0,1], X)$ and equipped with the sup distance. Notice that if (X, d) is complete and separable, then $C([0,1], X)$ is complete and separable as well. For $t \in [0,1]$ we denote by $e_t : C([0,1], X) \mapsto X$ the *evaluation map* defined by

$$e_t(\gamma) := \gamma_t, \qquad \forall \gamma \in C([0,1], X).$$

For $t, s \in [0,1]$ the map restr_t^s from $C([0,1], X)$ to itself is given by

$$(\mathrm{restr}_t^s \gamma)_r := \gamma_{t+r(s-t)}, \qquad \forall \gamma \in C([0,1], X).$$

The set of Borel probability measures on X is denoted by $\mathcal{P}(X)$ and $\mathcal{P}_2(X) \subset \mathcal{P}(X)$ is the space of probability measures with finite second moment. We equip $\mathcal{P}_2(X)$ with the quadratic transportation distance $\mathcal{W}_2$ defined by

$$\mathcal{W}_2^2(\mu, \nu) := \inf \int \mathrm{d}^2(x, y)\,\mathrm{d}\boldsymbol{\gamma}(x, y), \tag{2.11}$$

the inf being taken among all $\boldsymbol{\gamma} \in \mathcal{P}(X^2)$ such that

$$\pi^1_\sharp \boldsymbol{\gamma} = \mu,$$
$$\pi^2_\sharp \boldsymbol{\gamma} = \nu.$$

Given $\varphi : X \mapsto \mathbb{R} \cup \{-\infty\}$ not identically $-\infty$ the c-transform $\varphi^c : X \mapsto \mathbb{R} \cup \{-\infty\}$ is defined by

$$\varphi^c(y) := \inf_{x \in X} \frac{\mathrm{d}^2(x,y)}{2} - \varphi(x).$$

φ is said c-concave provided it is not identically $-\infty$ and $\varphi = \psi^c$ for some $\psi : X \mapsto \mathbb{R} \cup \{-\infty\}$. Equivalently, φ is c-concave if it is not identically $-\infty$ and $\varphi^{cc} = \varphi$. Given a c-concave function φ, its c-superdifferential $\partial^c\varphi \subset X^2$ is defined as the set of (x,y) such that

$$\varphi(x) + \varphi^c(y) = \frac{\mathrm{d}^2(x,y)}{2},$$

and for $x \in X$ the set $\partial^c\varphi(x)$ is the set of y's such that $(x,y) \in \partial^c\varphi$. Notice that for general $(x,y) \in X^2$ we have $\varphi(x) + \varphi^c(y) \le \frac{\mathrm{d}^2(x,y)}{2}$, thus $y \in \partial^c\varphi(x)$ can be equivalently formulated as

$$\varphi(z) - \varphi(x) \le \frac{\mathrm{d}^2(z,y)}{2} - \frac{\mathrm{d}^2(x,y)}{2}, \qquad \forall z \in X.$$

It turns out that for $\mu, \nu \in \mathscr{P}_2(X)$ the distance $\mathcal{W}_2(\mu,\nu)$ can be found as maximization of the dual problem of the optimal transport (2.11):

$$\frac{1}{2}\mathcal{W}_2^2(\mu,\nu) = \sup \int \varphi \,\mathrm{d}\mu + \int \varphi^c \,\mathrm{d}\nu, \tag{2.12}$$

the sup being taken among all c-concave functions φ. Notice that the integrals in the right hand side are well posed because for φ c-concave and $\mu, \nu \in \mathscr{P}_2(X)$ we always have $\max\{\varphi, 0\} \in L^1(\mu)$ and $\max\{\varphi^c, 0\} \in L^1(\nu)$. The sup is always achieved and any maximizing φ is called Kantorovich potential from μ to ν. For any Kantorovich potential we have in particular $\varphi \in L^1(\mu)$ and $\varphi^c \in L^1(\nu)$. Equivalently, the sup in (2.12) can be taken among all $\varphi : X \mapsto \mathbb{R}$ Lipschitz and bounded.

We shall make frequently use of the following superposition principle, proved in [33] (see also the original argument in the Euclidean framework [5]):

Proposition 2.1. *Let $\{\mu_t\}_t \subset \mathscr{P}_2(X)$ be a 2-a.c. curve w.r.t. $\mathcal{W}_2$. Then there exists $\pi \in \mathscr{P}(C([0,1],X))$ such that*

$$(e_t)_\sharp\pi = \mu_t, \qquad \forall t \in [0,1],$$
$$\int_0^1 |\dot\mu_t|^2 \,\mathrm{d}t = \iint_0^1 |\dot\gamma_t|^2 \,\mathrm{d}t \,\mathrm{d}\pi(\gamma),$$

and in particular π is concentrated on the set of 2-a.c. curves on X. For any such π we also have

$$|\dot\mu_t|^2 = \int |\dot\gamma_t|^2 \,\mathrm{d}\pi(\gamma), \qquad a.e.\ t \in [0,1].$$

Any plan π associated to the curve $\{\mu_t\}_t$ as in the above proposition will be called *lifting* of $\{\mu_t\}_t$.

2.2.2 Metric measure spaces and Sobolev functions

Spaces of interest for this paper are metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$ which will always be assumed to satisfy:

- (X, d) is complete and separable,
- the measure $\mathfrak{m}$ is a non-negative and non-zero Radon measure on X.

In this paper, for the abbreviation we will not distinguish $X, (X, \mathrm{d})$ or $(X, \mathrm{d}, \mathfrak{m})$ when no ambiguities exist. For example, we write $S^2(X)$ instead of $S^2(X, \mathrm{d}, \mathfrak{m})$.

Given a curve $\{\mu_t\}_t \subset \mathcal{P}(X)$ we shall say that it has *bounded compression* provided there is $C > 0$ such that $\mu_t \leq C\mathfrak{m}$ for every $t \in [0, 1]$. Similarly, given $\pi \in \mathcal{P}(C([0, 1], X))$ we shall say that it has bounded compression provided $t \mapsto \mu_t := (e_t)_\sharp \pi$ has bounded compression.

We shall now recall the definition of Sobolev functions 'having distributional differential in $L^2(X, \mathfrak{m})$'. The definition we adopt comes from [24] which in turn is a reformulation of the one proposed in [8]. For the proof that this approach produces the same concept as the one discussed in [30] and its references, see [8].

Definition 2.2 (Test plans)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space and $\pi \in \mathcal{P}(C([0, 1], X))$. We say that π is a test plan provided it has bounded compression and

$$\iint_0^1 |\dot{\gamma}_t|^2 \,\mathrm{d}t \,\mathrm{d}\pi(\gamma) < +\infty.$$

Definition 2.3 (The Sobolev class $S^2(X)$)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space. The Sobolev class $S^2(X) = S^2(X, \mathrm{d}, \mathfrak{m})$ is the space of all Borel functions $f : X \mapsto \mathbb{R}$ such that there exists a function $G \in L^2(X, \mathfrak{m})$, $G \geq 0$ such that:

$$\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi(\gamma) \leq \iint_0^1 G(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t \,\mathrm{d}\pi(\gamma),$$

for every test plan π. In this case, G is called a weak upper gradient of f.

It can be proved that for $f \in S^2(X)$ there exists a minimal, in the $\mathfrak{m}$-a.e. sense, weak upper gradient: we shall denote it by $|\mathrm{D}f|$.

Basic calculus rules for $|\mathrm{D}f|$ are the following, all the expressions being intended $\mathfrak{m}$-a.e.:

Locality For every $f, g \in S^2(X)$ we have

$$|\mathrm{D}f| = 0, \qquad \text{on } f^{-1}(N), \qquad \forall N \subset \mathbb{R}, \text{ Borel with } \mathcal{L}^1(N) = 0, \tag{2.13}$$

$$|\mathrm{D}f| = |\mathrm{D}g|, \qquad \mathfrak{m} - a.e. \text{ on } \{f = g\}. \tag{2.14}$$

Weak gradients and local Lipschitz constants. For any $f : X \mapsto \mathbb{R}$ locally Lipschitz it holds

$$|\mathrm{D}f| \leq \mathrm{lip}(f). \tag{2.15}$$

Vector space structure. $S^2(X)$ is a vector space and for every $f, g \in S^2(X)$, $\alpha, \beta \in \mathbb{R}$ we have

$$|\mathrm{D}(\alpha f + \beta g)| \leq |\alpha||\mathrm{D}f| + |\beta||\mathrm{D}g|. \tag{2.16}$$

Algebra structure. $L^\infty \cap S^2(X)$ is an algebra and for every $f, g \in L^\infty \cap S^2(X)$ we have

$$|\mathrm{D}(fg)| \leq |f||\mathrm{D}g| + |g||\mathrm{D}f|. \tag{2.17}$$

Similarly, if $f \in S^2(X)$ and g is Lipschitz and bounded, then $fg \in S^2(X)$ and the bound (2.17) holds.

Chain rule. Let $f \in S^2(X)$ and $\varphi : \mathbb{R} \mapsto \mathbb{R}$ Lipschitz. Then $\varphi \circ f \in S^2(X)$ and

$$|\mathrm{D}(\varphi \circ f)| = |\varphi'| \circ f |\mathrm{D}f|, \tag{2.18}$$

where $|\varphi'| \circ f$ is defined arbitrarily at points where φ is not differentiable (observe that the identity (2.13) ensures that on $f^{-1}(\mathcal{N})$ both $|\mathrm{D}(\varphi \circ f)|$ and $|\mathrm{D}f|$ are 0 $\mathfrak{m}$-a.e., $\mathcal{N}$ being the negligible set of points of non-differentiability of φ).

We equip $S^2(X)$ with the seminorm $\|f\|_{S^2} := \||\mathrm{D}f|\|_{L^2(X)}$. In the following, we will sometimes need to work with spaces $(X, \mathrm{d}, \mathfrak{m})$ such that $S^2(X)$ is separable, we thus recall the following general criterion:

Proposition 2.4. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space where $\mathfrak{m}$ is locally bounded. If the the Sobolev space $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ defined as $L^2 \cap S^2(X)$ equipped with the norm $\|f\|^2_{W^{1,2}} := \|f\|^2_{L^2} + \|f\|^2_{S^2}$ is reflexive, then $S^2(X)$ is separable.*

In particular, let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space satisfying one of the following properties below:

i) (X, d) is doubling, i.e. there is $N \in \mathbb{N}$ such that for any $r > 0$ any ball of radius $2r$ can be covered by N balls of radius r. And $\mathfrak{m}$ is locally bounded.

ii) *The seminorm* $\|\cdot\|_{S^2}$ *satisfies the parallelogram rule, and* $\mathfrak{m}$ *gives finite mass to bounded sets.*

We know $S^2(X)$ *is separable.*

Proof. In [1] it has been proved that if $W^{1,2}(X)$ is reflexive, then it is separable.

Thus to conclude it is sufficient to show that if $W^{1,2}(X)$ is separable and $\mathfrak{m}$ gives finite mass to bounded sets (this being trivially true in the case (i)), then $S^2(X)$ is separable as well. To this aim, let $f \in S^2(X)$, consider the truncated functions $f_n := \min\{n, \max\{-n, f\}\}$ and notice that thanks to (2.40) we have $\|f_n - f\|_{S^2} \to 0$ as $n \to \infty$. Thus we can reduce to consider the case of $f \in L^\infty \cap S^2(X)$. Let $B_n \subset X$ be bounded nondecreasing sequence of sets covering X, and for each $n \in \mathbb{N}$, $\chi_N : X \mapsto [0,1]$ a 1-Lipschitz function with bounded support and identically 1 on B_n. For $f \in L^\infty \cap S^2(X)$, by (2.17) we have $f\chi_n \in L^\infty \cap S^2(X)$ as well and furthermore $\mathrm{supp}(\chi_n f)$ is bounded. Given that $\mathfrak{m}$ gives finite mass to bounded sets we deduce that $f\chi_n \in W^{1,2}(X)$, and the locality property (2.14) ensures that $\|\chi_n f - f\|_{S^2} \to 0$ as $n \to \infty$.

At last, still in [1], it has been shown that if (X, d) is doubling, then $W^{1,2}(X)$ is reflexive. On the other hand, if (ii) holds, then it is obvious that $W^{1,2}(X)$ is Hilbert, and hence reflexive. Therefore $S^2(X)$ is separable if (i) or (ii) holds. □

2.2.3 Hopf-Lax formula and Hamilton-Jacobi equation

Here we recall the main properties of the Hopf-Lax formula and its link with the Hamilton-Jacobi equation in a metric setting. For a proof of these results see [8].

Definition 2.5 (Hopf-Lax formula)**.** Given $f : X \mapsto \mathbb{R}$ a function and $t > 0$ we define $Q_t f : X \mapsto \mathbb{R} \cup \{-\infty\}$ as

$$Q_t f(x) := \inf_{y \in X} f(y) + \frac{\mathrm{d}^2(x,y)}{2}.$$

We also put $Q_0 f := f$.

Proposition 2.6 (Basic properties of the Hopf-Lax formula)**.** *Let* $f : X \mapsto \mathbb{R}$ *be Lipschitz and bounded. Then the following hold.*

i) *For every* $t \geq 0$ *we have* $\mathrm{Lip}(Q_t f) \leq 2\,\mathrm{Lip}(f)$.

ii) For every $x \in X$ the map $[0,\infty) \ni t \mapsto Q_t f(x)$ is continuous, locally semiconcave on $(0,\infty)$ and the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t f(x) + \frac{\mathrm{lip}(Q_t f)^2(x)}{2} \leq 0,$$

holds for every $t \in (0,\infty)$ with at most a countable number of exceptions.

iii) The map $(0,\infty) \times X \ni (t,x) \mapsto \mathrm{lip}(Q_t f)(x)$ is upper-semicontinuous.

2.3 The continuity equation $\partial_t \mu_t = L_t$

2.3.1 Some definitions and conventions

Let $\mu \in \mathcal{P}_2(X)$ be such that $\mu \leq C\mathfrak{m}$ for some $C > 0$. We define the seminorm $\|\cdot\|_\mu$ on $S^2(X)$ as

$$\|f\|_\mu^2 := \int |\mathrm{D}f|^2 \, \mathrm{d}\mu.$$

Definition 2.7 (The cotangent space $\mathrm{CoTan}_\mu(X)$)**.** For $\mu \in \mathcal{P}_2(X)$ with $\mu \leq C\mathfrak{m}$ for some $C > 0$ consider the quotient space $S^2(X)/\sim_\mu$, where $f \sim_\mu g$ if $\|f - g\|_\mu = 0$.

The cotangent space $\mathrm{CoTan}_\mu(X)$ is then defined as the completion of $S^2(X)/\sim_\mu$ w.r.t. its natural norm. The norm on $\mathrm{CoTan}_\mu(X)$ will still be denoted by $\|\cdot\|_\mu$.

Given a linear map $L : S^2(X) \mapsto \mathbb{R}$ and μ as above, we denote by $\|L\|_\mu^*$ the quantity given by

$$\frac{1}{2}(\|L\|_\mu^*)^2 := \sup_{f \in S^2(X)} L(f) - \frac{1}{2}\|f\|_\mu^2.$$

Linear operators $L : S^2(X) \mapsto \mathbb{R}$ such that $\|L\|_\mu^* < \infty$ are in 1-1 correspondence with elements of the dual of $\mathrm{CoTan}_\mu(X)$. Abusing a bit the notation, we will often identify such operators L with the induced linear mapping on $\mathrm{CoTan}_\mu(X)$.

2.3.2 A localization argument

In this section $\{\mu_t\}_t \subset \mathcal{P}_2(X)$ is a given $\mathcal{W}_2$-continuous curve with bounded compression and we consider a functional $L : S^2(X) \mapsto L^1(0,1)$ satisfying the inequality

$$\int_t^s L(f)(r)\, dr \leq \sqrt{\int_t^s G_r^2 \, \mathrm{d}r} \sqrt{\int_t^s \|f\|_{\mu_r}^2 \, \mathrm{d}r}$$

with $G \in L^2(0,1)$, for every $f \in S^2(X)$ and $t, s \in [0,1]$, $t < s$. The question we address is up to what extent we can deduce that for such L there are operators $L_t : S^2(X) \mapsto \mathbb{R}$

such that $L(f)(t) = L_t(f)$ for $\mathcal{L}^1$-a.e. $t \in [0,1]$. We will see in a moment that this is always the case in an appropriate sense, but to deal with the case of $S^2(X)$ non separable we need to pay some attention to the set of Lebesgue points of $L(f)$.

Thus for given $g \in L^1(0,1)$ we shall denote by $\mathrm{Leb}(g) \subset (0,1)$ the set of t's such that the limit of

$$\fint_{t-\varepsilon}^{t+\varepsilon} g_s \, \mathrm{d}s,$$

as $\varepsilon \downarrow 0$ exists and is finite. Clearly the set $\mathrm{Leb}(g)$ contains all the Lebesgue points of any representative of g and in particular we have $\mathcal{L}^1(\mathrm{Leb}(g)) = 1$. We shall denote its value by $\bar{g}$, so that $\bar{g} : \mathrm{Leb}(g) \mapsto \mathbb{R}$ is a well chosen representative of g everywhere defined on $\mathrm{Leb}(g)$.

It is obvious that for $t \in \mathrm{Leb}(g)$, we have:

$$\overline{\lim_{\varepsilon_1,\varepsilon_2 \downarrow 0}} \fint_{t-\varepsilon_1}^{t+\varepsilon_1} \fint_{t-\varepsilon_2}^{t+\varepsilon_2} |g_s - g_r| \, \mathrm{d}s \, \mathrm{d}r = 0.$$

We then have the following result.

Lemma 2.8. *Let $\{\mu_t\}_t \subset \mathcal{P}_2(X)$ be a $\mathcal{W}_2$-continuous curve of bounded compression and $L : S^2(X) \mapsto L^1(0,1)$ be a linear map such that for some $G \in L^2(0,1)$ the inequality*

$$\int_t^s L(f)(r) \, dr \leq \sqrt{\int_t^s G_r^2 \, \mathrm{d}r} \sqrt{\int_t^s \|f\|_{\mu_r}^2 \, \mathrm{d}r}, \qquad \forall t, s \in [0,1], \ t < s, \ \forall f \in S^2(X), \tag{2.19}$$

holds.

Then there exists a family $\{L_t\}_{t \in [0,1]}$ of maps from $S^2(X)$ to $\mathbb{R}$ such that for any $f \in S^2(X)$ we have

$$L(f)(t) = L_t(f), \qquad a.e. \ t \in [0,1], \tag{2.20}$$

$$|L_t(f)| \leq |G_t| \|f\|_{\mu_t}, \qquad a.e. \ t \in [0,1]. \tag{2.21}$$

Remark 2.9. As a direct consequence of (2.20), if $\{\tilde{L}_t\}_{t \in [0,1]}$ ia another family of maps satisfies the above, then for every $f \in S^2(X)$ we have $L_t(f) = \tilde{L}_t(f)$ for a.e. $t \in [0,1]$. Thus L is unique as a map from $S^2(X)$ to $L^1(0,1)$.

Proof. For $f \in S^2(X)$ consider the set $\mathrm{Leb}(L(f))$ and for $t \in (0,1)$ let $V_t \subset S^2(X)$ be the set of f's in $S^2(X)$ such that $t \in \mathrm{Leb}(L(f))$. The trivial inclusion

$$\mathrm{Leb}(\alpha_1 g_1 + \alpha_2 g_2) \supset \mathrm{Leb}(g_1) \cap \mathrm{Leb}(g_2),$$

valid for any $g_1, g_2 \in L^1(0,1)$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ and the linearity of L grant that V_t is a vector space for every $t \in (0,1)$.

The $\mathcal{W}_2$-continuity of $\{\mu_t\}_t$ grants in particular continuity w.r.t. convergence in duality with $C_b(X)$ and the further assumption that $\mu_t \leq C\mathfrak{m}$ for any $t \in [0,1]$ ensures continuity w.r.t. convergence in duality with $L^1(X, \mathfrak{m})$. Thus the map $t \mapsto \int |\mathrm{D}f|^2 \,\mathrm{d}\mu_t$ is continuous for any $f \in S^2(X)$. Hence from inequality (2.19) we deduce that for any $f \in S^2(X)$ it holds

$$|\overline{L(f)}(t)| \leq \sqrt{\overline{G_t^2}}\, \|f\|_{\mu_t}, \qquad \forall t \in \mathrm{Leb}(L(f)) \cap \mathrm{Leb}(G^2),$$

which we can rewrite as: for any $t \in \mathrm{Leb}(G^2)$ it holds

$$|\overline{L(f)}(t)| \leq \sqrt{\overline{G_t^2}}\, \|f\|_{\mu_t}, \qquad \forall f \in V_t.$$

In other words, for any $t \in \mathrm{Leb}(G^2)$ the map $V_t \ni f \mapsto L_t(f) := \overline{L(f)}(t)$ is a well defined linear map from V_t to $\mathbb{R}$ with norm bounded by $\sqrt{\overline{G_t^2}}$.

By Hahn-Banach we can extend this map to a map from S^2 to $\mathbb{R}$ with norm bounded by $G(t)$. Noticing that by construction we have $f \in \bar{V}_t$ for a.e. $t \in [0,1]$ for any $f \in S^2$, the family of maps $\bar{L}_t$ fulfill the thesis. To conclude notice that trivially it holds $\sqrt{\overline{G_t^2}} = |G_t|$ for $\mathcal{L}^1$-a.e. t. □

2.3.3 Main theorem

We start giving the definition of 'distributional' solutions of the continuity equation in our setting:

Definition 2.10 (Solutions of $\partial_t \mu_t = L_t$)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space, $\{\mu_t\}_t \subset \mathcal{P}_2(X)$ a $\mathcal{W}_2$-continuous curve with bounded compression and $\{L_t\}_{t \in [0,1]}$ a family of maps from $S^2(X)$ to $\mathbb{R}$.

We say that $\{\mu_t\}_t$ solves the continuity equation

$$\partial_t \mu_t = L_t, \tag{2.22}$$

provided:

i) for every $f \in S^2(X)$ the map $t \mapsto L_t(f)$ is measurable and the map $N : [0,1] \mapsto [0, \infty]$ defined by
$$\frac{1}{2} N_t^2 := \operatorname*{ess\,sup}_{f \in S^2(X)} L_t(f) - \frac{1}{2}\|f\|^2_{\mu_t}, \tag{2.23}$$

belongs to $L^2(0,1)$, i.e. for any f, $\frac{1}{2}N_t^2 \geq L_t(f) - \frac{1}{2}\|f\|_{\mu_t}^2$ a.e. t and for any other $\bar{N}_t$ satisfying this property, we have: $N_t \leq \bar{N}_t$ for a.e. t.

ii) for every $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f \, \mathrm{d}\mu_t$ is in absolutely continuous and the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \, \mathrm{d}\mu_t = L_t(f),$$

holds for a.e. t.

Our main result is that for curves of bounded compression, the continuity equation characterizes 2-absolute continuity.

Theorem 2.11. *Let $\{\mu_t\}_t \subset P(X)$ be a $\mathcal{W}_2$-continuous curve with bounded compression. Then the following are equivalent.*

i) $\{\mu_t\}_t$ is 2-absolutely continuous w.r.t. $\mathcal{W}_2$.

ii) There is a family of maps $\{L_t\}_{t\in[0,1]}$ from $S^2(X)$ to $\mathbb{R}$ such that $\{\mu_t\}_t$ solves the continuity equation (2.22).

Finally, we have

$$N_t = |\dot{\mu}_t|, \qquad a.e.\ t \in [0,1].$$

Remark 2.12. It is obvious that if $\{\tilde{L}_t\}_{t\in[0,1]}$ is another family of maps such that $\{\mu_t\}_t$ solves $\partial_t \mu_t = \tilde{L}_t$, then for every $f \in L^1 \cap S^2(X)$ we have

$$L_t(f) = \tilde{L}_t(f), \qquad a.e.\ t \in [0,1],$$

thus $\{L_t\}_{t\in[0,1]}$ is unique as a map from $S^2(X)$ to $L^1(0,1)$.

Proof. (**i**) $\Rightarrow$ (**ii**) Let π be a lifting of $\{\mu_t\}_t$ and notice that π is a test plan. Hence for $f \in L^1 \cap S^2(X)$ we have

$$\begin{aligned} \left| \int f \, \mathrm{d}\mu_s - \int f \, \mathrm{d}\mu_t \right| &\leq \int |f(\gamma_s) - f(\gamma_t)| \, \mathrm{d}\pi(\gamma) \leq \iint_t^s |\mathrm{D}f|(\gamma_r)|\dot{\gamma}_r| \, \mathrm{d}r \, \mathrm{d}\pi(\gamma) \\ &\leq \sqrt{\int_t^s \int |\mathrm{D}f|^2 \, \mathrm{d}\mu_r \, \mathrm{d}r} \sqrt{\int_t^s \int |\dot{\gamma}_r|^2 \, \mathrm{d}\pi(\gamma) \, \mathrm{d}r}. \end{aligned} \tag{2.24}$$

Taking into account that $\int |\mathrm{D}f|^2 \, \mathrm{d}\mu_r \leq C \int |\mathrm{D}f|^2 \, \mathrm{d}\mathfrak{m}$ for every $t \in [0,1]$, this shows that $t \mapsto \int f \, \mathrm{d}\mu_t$ is absolutely continuous.

Define $L : S^2(X) \mapsto L^1(0,1)$ by $L(f)(t) := \partial_t \int f \, \mathrm{d}\mu_t$ and notice that the bound (4.3) gives

$$\left| \int_t^s L(f)(r)\,\mathrm{d}r \right| \leq \sqrt{\int_t^s \int |\mathrm{D}f|^2 \,\mathrm{d}\mu_r \,\mathrm{d}r} \sqrt{\int_t^s G_r^2 \,\mathrm{d}r}, \qquad \forall t,s \in [0,1],\ t<s,$$

for $G_t := |\dot{\mu}_t| = \sqrt{\int |\dot{\gamma}_r|^2 \,\mathrm{d}\pi(\gamma)} \in L^2(0,1)$. Hence we can apply Lemma 2.8 and deduce from (2.21) that for every $f \in S^2(X)$ we have

$$L_t(f) - \frac{1}{2}\|f\|^2_{\mu_t} \leq \frac{1}{2}|\dot{\mu}_t|^2, \qquad a.e.\ t \in [0,1].$$

By the definition (2.23), this latter bound is equivalent to $N_t \leq |\dot{\mu}_t|$ for a.e. $t \in [0,1]$.

(ii) $\Rightarrow$ **(i)** To get the result it is sufficient to prove that

$$\mathcal{W}_2^2(\mu_t,\mu_s) \leq |s-t| \int_t^s N_r^2 \,\mathrm{d}r, \qquad \forall t,s \in [0,1],\ t<s.$$

We shall prove this bound for $t=0$ and $s=1$ only, the general case following by a simple rescaling argument. Recalling that

$$\frac{1}{2}\mathcal{W}_2^2(\mu_0,\mu_1) = \sup_{\psi} \int \psi \,\mathrm{d}\mu_0 + \int \psi^c \,\mathrm{d}\mu_1 = \sup_{\varphi} \int Q_1\varphi \,\mathrm{d}\mu_1 - \int \varphi \,\mathrm{d}\mu_0,$$

the sup being taken among all Lipschitz and bounded ψ,φ, to get the claim it is sufficient to prove that

$$\int Q_1\varphi \,\mathrm{d}\mu_1 - \int \varphi \,\mathrm{d}\mu_0 \leq \frac{1}{2}\int_0^1 N_t^2 \,\mathrm{d}t, \tag{2.25}$$

for any Lipschitz and bounded $\varphi : X \mapsto \mathbb{R}$. Fix such φ and notice that

$$\int Q_1\varphi \,\mathrm{d}\mu_1 - \int \varphi \,\mathrm{d}\mu_0 = \lim_{n\to\infty} \left\{ \sum_{i=0}^{n-1} \int (Q_{\frac{i+1}{n}}\varphi - Q_{\frac{i}{n}}\varphi) \,\mathrm{d}\mu_{\frac{i+1}{n}} + \int Q_{\frac{i}{n}}\varphi \,\mathrm{d}(\mu_{\frac{i+1}{n}} - \mu_{\frac{i}{n}}) \right\}. \tag{2.26}$$

Recalling point (ii) of Proposition 2.6 we have

$$\sum_{i=0}^{n-1} \int (Q_{\frac{i+1}{n}}\varphi - Q_{\frac{i}{n}}\varphi) \,\mathrm{d}\mu_{\frac{i}{n}} \leq \sum_{i=0}^{n-1} \int \int_{\frac{i}{n}}^{\frac{i+1}{n}} -\frac{\mathrm{lip}(Q_t\varphi)^2}{2} \,\mathrm{d}t \,\mathrm{d}\mu_{\frac{i}{n}} = \int_{X\times[0,1]} -\frac{\mathrm{lip}(Q_t\varphi)^2(x)}{2} \,\mathrm{d}\mu_n(x,t).$$

where $\mu_n := \sum_{i=0}^{n-1} \mu_{\frac{i}{n}} \times \mathcal{L}^1|_{[\frac{i}{n},\frac{i+1}{n}]}$. The continuity of $\{\mu_t\}_t$ easily yields that (μ_n) converges to $\mu := \mathrm{d}\mu_t(x) \otimes \mathrm{d}t$ in duality with $C_b(X\times[0,1])$. Furthermore, the assumption $\mu_t \leq C\mathfrak{m}$ for every $t \in [0,1]$ yields $\mu_n \leq C\mathfrak{m} \times \mathcal{L}^1$ for every $n \in \mathbb{N}$ and thus by the Dunfort-Pettis theorem (see for instance Theorem 4.7.20 in [18]) we deduce that (μ_n) converges to $\mu \in \mathcal{P}(X\times[0,1])$, $\mathrm{d}\mu := \mathrm{d}\mu_t \otimes \mathrm{d}t$, in duality with $L^\infty(X\times[0,1])$. Being

$(t,x)\mapsto\frac{\operatorname{lip}(Q_t\varphi)^2(x)}{2}$ bounded (point (i) of Proposition 2.6), we deduce that

$$\varlimsup_{n\to\infty}\sum_{i=0}^{n-1}\int(Q_{\frac{i+1}{n}}\varphi-Q_{\frac{i}{n}}\varphi)\,\mathrm{d}\mu_{\frac{i}{n}}\le\iint_0^1-\frac{\operatorname{lip}(Q_t\varphi)^2(x)}{2}\,\mathrm{d}\mu_t\,\mathrm{d}t. \tag{2.27}$$

On the other hand we have

$$\begin{aligned}\sum_{i=0}^{n-1}\int Q_{\frac{i}{n}}\varphi\,\mathrm{d}(\mu_{\frac{i+1}{n}}-\mu_{\frac{i}{n}})&=\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}}L_s(Q_{\frac{i}{n}}\varphi)\,\mathrm{d}s\\&\le\sum_{i=0}^{n-1}\frac{1}{2}\int_{\frac{i}{n}}^{\frac{i+1}{n}}N_s^2\,\mathrm{d}s+\sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}}\int\frac{|\mathrm{D}Q_{\frac{i}{n}}\varphi|^2}{2}\,\mathrm{d}\mu_s\,\mathrm{d}s\\&\le\frac{1}{2}\int_0^1N_t^2\,\mathrm{d}t+\int_{X\times[0,1]}f_n(t,x)\,\mathrm{d}\mu,\end{aligned}$$

where $f_n(t,x):=\frac{\operatorname{lip}(Q_{\frac{i}{n}}\varphi)^2(x)}{2}$ for $t\in[\frac{i}{n},\frac{i+1}{n})$ and $\mathrm{d}\mu(t,x):=\mathrm{d}\mu_t(x)\otimes\mathrm{d}t$. Recall that by points $(i),(iii)$ of Proposition 2.6 we have that the f_n's are equibounded and satisfy $\varlimsup_n f_n(t,x)\le f(t,x):=\frac{\operatorname{lip}(Q_t\varphi)^2(x)}{2}$, thus Fatou's lemma gives

$$\varlimsup_{n\to\infty}\int_{X\times[0,1]}f_n(t,x)\,\mathrm{d}\mu\le\int_{X\times[0,1]}f(t,x)\,\mathrm{d}\mu,$$

and therefore

$$\varlimsup_{n\to\infty}\sum_{i=0}^{n-1}\int Q_{\frac{i+1}{n}}\varphi\,\mathrm{d}(\mu_{\frac{i+1}{n}}-\mu_{\frac{i}{n}})\le\frac{1}{2}\int_0^1N_t^2\,\mathrm{d}t+\iint_0^1\frac{\operatorname{lip}(Q_t\varphi)^2}{2}\,\mathrm{d}\mu_t\,\mathrm{d}t \tag{2.28}$$

The bounds (2.27) and (2.28) together with (2.26) give (2.25) and the thesis. □

If we know that $S^2(X)$ is separable, the result is slightly stronger, as a better description of the operators $\{L_t\}$ is possible, as shown by the following statement.

Proposition 2.13. *Let $\{\mu_t\}_t\subset\mathcal{P}_2(X)$ be a 2-absolutely continuous curve w.r.t. $\mathcal{W}_2$ of bounded compression. Assume furthermore that $S^2(X)$ is separable. Then there exists a $\mathcal{L}^1$-negligible set $\mathcal{N}\subset[0,1]$ and, for every $t\in[0,1]\setminus\mathcal{N}$, a linear map $L_t:S^2(X)\mapsto\mathbb{R}$ such that:*

i) every $t\in[0,1]\setminus\mathcal{N}$ is a Lebesgue point of $s\mapsto|\dot\mu_s|^2$, the metric speed $|\dot\mu_s|$ exists at $s=t$ and we have $|\dot\mu_t|=\|L_t\|^*_{\mu_t}$, *ii) for every $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f \, \mathrm{d}\mu_t$ is absolutely continuous, differentiable at every $t \in [0,1] \setminus \mathcal{L}$ and its derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \, \mathrm{d}\mu_t = L_t(f), \qquad \forall t \in [0,1] \setminus \mathcal{N}.$$

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subset S^2(X)$ be a countable dense set and $\mathcal{N} \subset [0,1]$ the set of t's such that either the metric speed $|\dot{\mu}_t|$ does not exist, or t is not a Lebesgue point of $s \mapsto |\dot{\mu}_s|^2$ or for some $n \in \mathbb{N}$ the map $s \mapsto \int f \, \mathrm{d}\mu_s$ is not differentiable at t. Then by Theorem 4.9 we know that $\mathcal{N}$ is negligible.

For $n \in \mathbb{N}$ and $t \in [0,1] \setminus \mathcal{N}$, inequality (4.3) gives, after a division for $|s-t|$ and a limit $s \to t$, the bound

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \int f_n \, \mathrm{d}\mu_t \right| \le |\dot{\mu}_t| \sqrt{\int |\mathrm{D} f_n|^2 \, \mathrm{d}\mu_t} \le C |\dot{\mu}_t| \sqrt{\int |\mathrm{D} f_n|^2 \, \mathfrak{m}}.$$

This means that the map $S^2(X) \ni f_n \mapsto \frac{\mathrm{d}}{\mathrm{d}t} \int f_n \, \mathrm{d}\mu_t$ can be uniquely extended to a linear operator L_t from $S^2(X)$ to $\mathbb{R}$ which satisfies $\|L_t\|^*_{\mu_t} \le |\dot{\mu}_t|$.

For $f \in L^1 \cap S^2(X)$ denote by $I_f : [0,1] \mapsto \mathbb{R}$ the function given by $I_f(t) := \int f \, \mathrm{d}(\mu_t - \mu_0)$. Then the map $f \mapsto I_f$ is clearly linear and satisfies

$$|I_f(t)| \le \int |f(\gamma_t) - f(\gamma_0)| \, \mathrm{d}\pi(\gamma) \le \iint_0^t |\mathrm{D} f|(\gamma_s) |\dot{\gamma}_s| \, \mathrm{d}s \, \mathrm{d}\pi(\gamma) \le \sqrt{C} \|f\|_{S^2} \sqrt{\iint_0^1 |\dot{\gamma}_t|^2 \, \mathrm{d}\pi(\gamma)}.$$

Hence given that we have $I_{f_n}(t) = \int_0^t L_t(f_n) \, \mathrm{d}t$ for every $n \in \mathbb{N}$ and that $\{f_n\}$ is dense in $L^1 \cap S^2(X)$ w.r.t. the (semi)distance of $S^2(X)$, from the bound $N_t \le |\dot{\mu}_t|$ for $\mathcal{L}^1$-a.e. t, we deduce that $I_f(t) = \int_0^t L_t(f) \, \mathrm{d}t$ for every $f \in L^1 \cap S^2(X)$ and every $t \in [0,1]$.

Along the same lines we have that

$$\begin{aligned} |I_f(s) - I_f(t)| &\le \int |f(\gamma_s) - f(\gamma_t)| \, \mathrm{d}\pi(\gamma) \le \iint_t^s |\mathrm{D} f|(\gamma_r) |\dot{\gamma}_r| \, \mathrm{d}r \, \mathrm{d}\pi(\gamma) \le \\ &\le \sqrt{C|s-t| \iint_t^s |\dot{\gamma}_s|^2 \, \mathrm{d}r \, \mathrm{d}\pi(\gamma)} \|f\|_{S^2} = \sqrt{C|s-t| \int_t^s |\dot{\mu}_r|^2 \, \mathrm{d}r} \|f\|_{S^2} \end{aligned}$$

and therefore for every $t \in [0,1]$ Lebesgue point of $s \mapsto |\dot{\mu}_s|^2$ and such that $|\dot{\mu}_t|$ exists we have

$$\overline{\lim_{s \to t}} \left| \frac{I_f(s) - I_f(t)}{s - t} \right| \le \sqrt{C} \|f\|_{S^2} |\dot{\mu}_t|.$$

Taking into account that, by construction, we have $\lim_{s \to t} \frac{I_{f_n}(s) - I_{f_n}(t)}{s-t} = L_t(f_n)$ for every $t \in [0,1] \setminus \mathcal{N}$ and the density of $\{f_n\}$, we deduce that $\lim_{s \to t} \frac{I_f(s) - I_f(t)}{s-t} = L_t(f)$ for every $f \in L^1 \cap S^2(X)$ and $t \in [0,1] \setminus \mathcal{N}$.

It remains to prove that $\|L_t\|^*_{\mu_t} = |\dot{\mu}_t|$ for $t \in [0,1] \setminus \mathcal{N}$. From Theorem 4.9 we know that $N_t = |\dot{\mu}_t|$ for $\mathcal{L}^1$-a.e. t and to conclude use the separability of $S^2(X)$ to get

$$\frac{1}{2}(\|L_t\|^*_{\mu_t})^2 = \sup_{n\in\mathbb{N}} L_t(f_n) - \frac{1}{2}\|f_n\|^2_{\mu_t} = \operatorname*{ess\,sup}_{f\in S^2(X)} L_t(f) - \frac{1}{2}\|f\|^2_{\mu_t} = \frac{1}{2}N_t^2, \qquad a.e.\ t,$$

so that up to enlarging $\mathcal{N}$ we get the thesis. □

2.3.4 Some consequences in terms of differential calculus

As discussed in [37], see also [5], the continuity equation plays a key role in developing a first order calculus on the space $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$. In this section, we show that the continuity equation plays a similar role on metric measure spaces, where no smooth structure is a priori given. The only technical difference one needs to pay attention to is the fact that only curves with bounded compression should be taken into account.

We start with the Benamou-Brenier formula. Recall that on $\mathbb{R}^d$, and more generally Riemannian/Finslerian manifolds, we have the identity

$$\mathcal{W}_2^2(\mu_0,\mu_1) = \inf \int_0^1 \int |v_t|^2 \,\mathrm{d}\mu_t \,\mathrm{d}t, \tag{2.29}$$

where the inf is taken among all 2-a.c. curves $\{\mu_t\}_t$ joining μ_0 to μ_1 and the v_t's are such that the continuity equation

$$\partial_t \mu_t + \nabla\cdot(v_t\mu_t) = 0, \tag{2.30}$$

holds. We want to investigate the validity of this formula in the metric-measure context. To this aim, notice that formula (2.29) expresses the fact that the distance $\mathcal{W}_2$ can be realized as inf of length of curves, where this length is measured in an appropriate way. Hence there is little hope to get an analogous of this formula on $(X, \mathrm{d}, \mathfrak{m})$ unless we require in advance that (X, d) is a length space. Furthermore, given that in the non-smooth case we are confined to work with curves with bounded compression, we need to enforce a length structure compatible with the measure $\mathfrak{m}$, thus we are led to the following definition:

Definition 2.14 (Measured-length spaces)**.** We say that $(X, \mathrm{d}, \mathfrak{m})$ is measured-length provided for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with bounded support and satisfying $\mu_0, \mu_1 \le C\mathfrak{m}$ for some $C > 0$ the distance $\mathcal{W}_2(\mu_0, \mu_1)$ can be realized as inf of length of absolutely continuous curves $\{\mu_t\}_t$ with bounded compression connecting μ_0 to μ_1.

On measured-length spaces we then have a natural analog of formula (2.29), which is in fact a direct consequence of Theorem 4.9:

Proposition 2.15 (Benamou-Brenier formula on metric measure spaces)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a measured-length space and $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with bounded support and satisfying $\mu_0, \mu_1 \leq C\mathfrak{m}$ for some $C > 0$.*

Then we have

$$W_2^2(\mu_0, \mu_1) = \inf \int_0^1 (\|L_t\|^*_{\mu_t})^2 \,\mathrm{d}t,$$

the inf *being taken among all 2-absolutely continuous curves $\{\mu_t\}_t$ with bounded compression joining μ_0 to μ_1 and the operators (L_t) are those associated to the curve via Theorem 4.9.*

Proof. By Theorem 4.9 we know that for a 2-absolutely continuous curve $\{\mu_t\}_t$ with bounded compression we have $|\dot{\mu}_t| = \|L_t\|^*_{\mu_t}$ for a.e. $t \in [0, 1]$, the operators $\{L_t\}$ being those associated to the curve via Theorem 4.9 itself. The conclusion then follows directly from the definition of measured-length space. □

We now discuss the formula for the derivative of $t \mapsto \frac{1}{2}W_2^2(\mu_t, \nu)$, where $\{\mu_t\}_t$ is a 2-a.c. curve with bounded compression. Recall that on the Euclidean setting we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu) = \int \nabla\varphi_t \cdot v_t \,\mathrm{d}\mu_t, \qquad a.e.\ t,$$

where φ_t is a Kantorovich potential from μ_t to ν for every $t \in [0, 1]$ and the vector fields (v_t) are such that the continuity equation (2.30) holds. Due to our interpretation of the continuity equation in the metric measure setting, we are therefore lead to guess that in the metric-measure setting we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu) = L_t(\varphi_t), \qquad a.e.\ t. \tag{2.31}$$

As we shall see in a moment (Proposition 2.17) this is actually the case in quite high generality, but before coming to the proof, we need to spend few words on how to interpret the right hand side of (2.31) because in general we don't have $\varphi_t \in S^2(X)$ so that a priory φ_t is outside the domain of definition of L_t. This can in fact be easily fixed by considering φ_t as element of $\mathrm{CoTan}_\mu(X)$, as defined in Section 2.3.1. This is the scope of the following lemma.

Lemma 2.16. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a m.m.s. such that $\mathfrak{m}$ gives finite mass to bounded sets and φ a c-concave function such that $\partial^c\varphi(x) \cap B \neq \emptyset$ for every $x \in X$ and some bounded set $B \subset X$. Define $\varphi_n := \min\{n, \varphi\}$.*

Then $\varphi_n \in S^2(X)$ *and* (φ_n) *is a Cauchy sequence w.r.t. the seminorm* $\|\cdot\|_\mu \to 0$ *for every* $\mu \in \mathcal{P}_2(X)$ *such that* $\mu \leq C\mathfrak{m}$ *for some* $C > 0$.

Proof. We first claim that $\sup_B \varphi^c < \infty$. Indeed, if not there is a sequence $(y_n) \subset B$ such that $\varphi^c(y_n) > n$ for every $n \in \mathbb{N}$. Hence for every $x \in X$ we would have

$$\varphi(x) \leq \inf_{n\in\mathbb{N}} \mathrm{d}^2(x, y_n) - \varphi^c(y_n) \leq \inf_{n\in\mathbb{N}} \frac{1}{2}\big(\mathrm{d}(x, B) + \mathrm{diam}(B)\big)^2 - n = -\infty,$$

contradicting the definition of c-concavity. Using the assumption we have

$$\varphi(x) = \inf_{y\in B} \frac{\mathrm{d}^2(x, y)}{2} - \varphi^c(y) \geq \frac{\mathrm{d}^2(x, B)}{2} - \sup_B \varphi^c. \tag{2.32}$$

This proves that φ is bounded from below and that it has bounded sublevels. Hence the truncated functions φ_n are constant outside a bounded set. Now let $x, x' \in X$ and $y \in \partial^c\varphi(x) \cap B$. Then we have

$$\varphi(x) - \varphi(x') \leq \frac{\mathrm{d}^2(x, y) - \mathrm{d}^2(x'y)}{2} \leq \mathrm{d}(x, x')\left(\frac{\mathrm{d}(x, B) + \mathrm{d}(x', B)}{2} + \mathrm{diam}(B)\right).$$

Inverting the roles of x, x' we deduce that φ is Lipschitz on bounded sets and the pointwise estimate

$$\mathrm{lip}(\varphi)(x) \leq \mathrm{d}(x, B) + \mathrm{diam}(B), \qquad \forall x \in X. \tag{2.33}$$

It follows that the φ_n's are Lipschitz and, using the fact that $\mathfrak{m}$ gives finite mass to bounded sets, that $\varphi_n \in S^2(X)$ for every $n \in \mathbb{N}$.

To conclude, notice that the bound (2.33) ensures that for any $\mu \in \mathcal{P}_2(X)$ we have $\mathrm{lip}(\varphi) \in L^2(\mu)$. Thus for μ such that $\mu \leq C\mathfrak{m}$ for some $C > 0$ we have $|\mathrm{D}\varphi| \leq \mathrm{lip}(\varphi)$ μ-a.e. and thus $|\mathrm{D}\varphi| \in L^2(\mu)$ as well. Now observe that

$$\|\varphi_m - \varphi_n\|_\mu^2 = \int_{\{\varphi_m \neq \varphi_n\}} |\mathrm{D}\varphi|^2 \, \mathrm{d}\mu,$$

and that the right hand side goes to 0 as $n, m \to \infty$, because by (2.32) we know that $\cup_n\{\varphi = \varphi_n\} = X$. □

Thanks to this lemma we can, and will, associate the Kantorovich potential φ as an element of $\mathrm{CoTan}_\mu(X)$: it is the limit of the equivalence classes of the truncated functions φ_n.

Recall that for $\mu, \nu \in \mathcal{P}_2(X)$, there always exists a Kantorovich potential φ from μ to ν such that

$$\varphi(x) = \inf_{y \in \mathrm{supp}(\nu)} \frac{\mathrm{d}^2(x,y)}{2} - \varphi^c(y), \qquad \forall x \in X, \tag{2.34}$$

hence if ν has bounded support, a potential satisfying the assumption of Lemma 2.16 above can always be found.

We can now state and prove the following result about the derivative of $\mathcal{W}_2^2(\cdot,\nu)$. It is worth noticing that formula (2.35) below holds even for spaces which are not length spaces.

Proposition 2.17 (Derivative of $\mathcal{W}_2^2(\cdot,\nu)$)**.** *Let $\{\mu_t\}_t \subset \mathcal{P}_2(X)$ be a 2-a.c. curve with bounded compression, $\nu \in \mathcal{P}_2(X)$ with bounded support and notice that $t \mapsto \frac{1}{2}\mathcal{W}_2^2(\mu_t,\nu)$ is absolutely continuous. Assume that $S^2(X)$ is separable and that $\mathfrak{m}$ gives finite mass to bounded sets. Then the for a.e. $t \in [0,1]$ the formula*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathcal{W}_2^2(\mu_t,\nu) = L_t(\varphi_t), \tag{2.35}$$

holds, where φ_t is any Kantorovich potential from μ_t to ν fulfilling the assumptions of Lemma 2.16.

Proof. Let $\mathcal{N} \subset [0,1]$ be the $\mathcal{L}^1$-negligible set given by Proposition 2.13. We shall prove formula (2.35) for every $t \in [0,1] \setminus \mathcal{N}$ such that $\frac{1}{2}\mathcal{W}_2^2(\mu_\cdot,\nu)$ is differentiable at t. Fix such t, let φ_t be as in the assumptions and notice that

$$\begin{aligned} \frac{1}{2}\mathcal{W}_2^2(\mu_t,\nu) &= \int \varphi_t \, d\mu_t + \int \varphi^c \, d\nu, \\ \frac{1}{2}\mathcal{W}_2^2(\mu_s,\nu) &\geq \int \varphi_t \, d\mu_s + \int \varphi^c \, d\nu, \qquad \forall s \in [0,1], \end{aligned}$$

and thus

$$\frac{\mathcal{W}_2^2(\mu_s,\nu) - \mathcal{W}_2^2(\mu_t,\nu)}{2} \geq \int \varphi_t \mathrm{d}(\mu_s - \mu_t).$$

Recall that $\max\{\varphi_t, 0\} \in L^1(\mu)$ for every $\mu \in \mathcal{P}_2(X)$ and that $\varphi_t \in L^1(\mu_t)$, so that the integral in the right hand side makes sense. Put $\varphi_{n,t} := \min\{n, \max\{-n, \varphi_t\}\}$ so that by Lemma 2.16 above we have $\varphi_{n,t} \in S^2(X)$ for every $n \in \mathbb{N}$ and $\|\varphi_{n,t} - \varphi_{m,t}\|_\mu \to 0$ as $n, m \to \infty$. For every $n \in \mathbb{N}$ we know that $\frac{\mathrm{d}}{\mathrm{d}s} \int \varphi_{n,t} \, \mathrm{d}\mu_s|_{s=t} = L_t(\varphi_{n,t})$ and by Lemma 2.16 we know that $L_t(\varphi_{n,t}) \to L_t(\varphi_t)$ as $n \to \infty$. To conclude it is sufficient to notice that for any lifting π of $\{\mu_t\}_t$ we have the bound

$$\begin{aligned}\left|\int(\varphi_{n,t}-\varphi_{m,t})\mathrm{d}\frac{\mu_s-\mu_t}{s-t}\right| &\leq \frac{1}{|s-t|}\int(\varphi_{n,t}-\varphi_{m,t})(\gamma_s)-(\varphi_{n,t}-\varphi_{m,t})(\gamma_t)\,\mathrm{d}\pi(\gamma)\\ &\leq \frac{1}{|s-t|}\iint_t^s|\mathrm{D}(\varphi_{n,t}-\varphi_{m,t})|(\gamma_r)|\dot\gamma_r|\,\mathrm{d}r\,\mathrm{d}\pi(\gamma)\\ &\leq \sqrt{⨍_t^s\|\mathrm{D}(\varphi_{n,t}-\varphi_{m,t})\|^2_{\mu_r}\,\mathrm{d}r}\sqrt{\int⨍_t^s|\dot\gamma_r|^2\,\mathrm{d}r\,\mathrm{d}\pi(\gamma)},\end{aligned}$$

and that the dominated convergence theorem ensures that $⨍_t^s\|\mathrm{D}(\varphi_{n,t}-\varphi_{m,t})\|^2_{\mu_r}\,\mathrm{d}r\to 0$ as $n,m\to\infty$. □

2.4 The continuity equation $\partial_t\mu_t+\nabla\cdot(\nabla\phi_t\mu_t)=0$

2.4.1 Preliminaries: duality between differentials and gradients

On Euclidean spaces it is often the case that the continuity equation (2.30) can be written as

$$\partial_t\mu_t+\nabla\cdot(\nabla\phi_t\mu_t)=0, \tag{2.36}$$

for some functions ϕ_t, so that the vector fields v_t can be represented as gradient of functions. In some sense, the 'optimal' velocity vector fields (i.e. those minimizing the $L^2(\mu_t)$-norm) can always be thought of as gradients, as they always belong to the closure of the space of gradients of smooth functions w.r.t. the $L^2(\mu_t)$-norm (i.e. they belong to the cotangent space $\mathrm{CoTan}_{\mu_t}(\mathbb{R}^d)$), see [5]. Yet, the process of taking completion in general destroys the property of being the gradient of a smooth/Sobolev functions, so that technically speaking general absolutely continuous curves solve (2.30) and only in some cases one can write it as in (2.36).

It is then the scope of this part of the paper to investigate how one can give a meaning to (2.36) in the non-smooth setting and which sort of information on the curve we can obtain from such 'PDE'. According to our interpretation of the continuity equation given in Theorem 4.9, the problem reduces to understand in what sense we can write $L_t(f)=\int\mathrm{D}f(\nabla\phi_t)\,\mathrm{d}\mu_t$, and thus ultimately to give a meaning to 'the differential of a function applied to the gradient of another function'. This has been the scope of [24], we recall here the main definitions and properties.

Definition 2.18 (The objects $\mathrm{D}^\pm f(\nabla g)$)**.** Let $(X,\mathrm{d},\mathfrak{m})$ be a m.m.s. and $f,g\in S^2(X)$.

The functions $\mathrm{D}^{\pm}f(\nabla g) : X \mapsto \mathbb{R}$ are $\mathfrak{m}$-a.e. well defined by

$$\mathrm{D}^{+}f(\nabla g) := \lim_{\varepsilon\downarrow 0} \frac{|\mathrm{D}(g+\varepsilon f)|^2 - |\mathrm{D}g|^2}{2\varepsilon},$$
$$\mathrm{D}^{-}f(\nabla g) := \lim_{\varepsilon\uparrow 0} \frac{|\mathrm{D}(g+\varepsilon f)|^2 - |\mathrm{D}g|^2}{2\varepsilon}.$$

It is immediate to check that for $\varepsilon_1 < \varepsilon_2$ we have

$$\frac{|\mathrm{D}(g+\varepsilon_1 f)|^2 - |\mathrm{D}g|^2}{2\varepsilon_1} \le \frac{|\mathrm{D}(g+\varepsilon_2 f)|^2 - |\mathrm{D}g|^2}{2\varepsilon_2}, \qquad \mathfrak{m} - a.e.,$$

so that the limits above can be replaced by $\inf_{\varepsilon>0}$ and $\sup_{\varepsilon<0}$ respectively.

Heuristically, we should think to $\mathrm{D}^{+}f(\nabla g)$ (resp. $\mathrm{D}^{-}f(\nabla g)$) as the maximal (resp. minimal) value of the differential of f applied to all possible gradients of g, see [24] for a discussion on this topic.

The basic algebraic calculus rules for $D^{\pm}f(\nabla g)$ are the following:

$$|\mathrm{D}^{\pm}(f_1 - f_2)(\nabla g)| \le |\mathrm{D}(f_1 - f_2)||\mathrm{D}g|, \tag{2.37}$$
$$\mathrm{D}^{-}f(\nabla g) \le \mathrm{D}^{+}f(\nabla g),$$
$$\mathrm{D}^{+}(-f)(\nabla g) = \mathrm{D}^{+}f(\nabla(-g)) = -\mathrm{D}^{-}f(\nabla g), \tag{2.38}$$
$$\mathrm{D}^{\pm}g(\nabla g) = |\mathrm{D}g|^2. \tag{2.39}$$

We also have natural chain rules: given $\varphi : \mathbb{R} \mapsto \mathbb{R}$ Lipschitz we have

$$\begin{aligned} \mathrm{D}^{\pm}(\varphi\circ f)(\nabla g) &= \varphi'\circ f\,\mathrm{D}^{\pm\mathrm{sign}\varphi'\circ f}f(\nabla g), \\ \mathrm{D}^{\pm}f(\nabla\varphi\circ g) &= \varphi'\circ g\,\mathrm{D}^{\pm\mathrm{sign}\varphi'\circ g}f(\nabla g), \end{aligned} \tag{2.40}$$

where $\varphi'\circ f$ (resp. $\varphi'\circ g$) are defined arbitrarily at those x's such that φ is not differentiable at $f(x)$ (resp. $g(x)$). In particular, $\mathrm{D}^{\pm}f(\nabla(\alpha g)) = \alpha\mathrm{D}^{\pm}f(\nabla g)$ for $\alpha > 0$.

Notice that as a consequence of the above we have that for given $g \in S^2(X)$ the map $S^2(X) \ni f \mapsto \mathrm{D}^{+}f(\nabla g)$ is $\mathfrak{m}$-a.e. convex, i.e.

$$\mathrm{D}^{+}((1-\lambda)f_1 + \lambda f_2)(\nabla g) \le (1-\lambda)\mathrm{D}^{+}f_1(\nabla g) + \lambda\mathrm{D}^{+}f_2(\nabla g), \qquad \mathfrak{m} - a.e., \tag{2.41}$$

for every $f_1, f_2 \in S^2(X)$, and $\lambda \in [0,1]$. Similarly, $f \mapsto \mathrm{D}^{-}f(\nabla g)$ is $\mathfrak{m}$-a.e. concave.

Furthermore, it is easy to see that for $g \in S^2(X)$ and π test plan we have

$$\varlimsup_{t\downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t}\,\mathrm{d}\pi(\gamma) \le \frac{1}{2}\int |\mathrm{D}g|^2(\gamma_0)\,\mathrm{d}\pi(\gamma) + \frac{1}{2}\varlimsup_{t\downarrow 0}\frac{1}{t}\iint_0^t |\dot\gamma_s|^2\,\mathrm{d}s\,\pi(\gamma). \tag{2.42}$$

We are therefore lead to the following definition:

Definition 2.19 (Plans representing gradients)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a m.m.s. $g \in S^2(X)$ and π a test plan. We say that π represents the gradient of g provided it is a test plan and we have

$$\varliminf_{t \downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t} \, \mathrm{d}\pi(\gamma) \geq \frac{1}{2} \int |\mathrm{D}g|^2(\gamma_0) \, \mathrm{d}\pi(\gamma) + \frac{1}{2} \varlimsup_{t \downarrow 0} \frac{1}{t} \iint_0^t |\dot{\gamma}_s|^2 \, \mathrm{d}s \, \pi(\gamma). \tag{2.43}$$

It is worth noticing that plans representing gradients exist in high generality (see [24]).

Differentiation along plans representing gradients is tightly linked to the object $D^{\pm} f(\nabla g)$ defined above: this is the content of the following simple but crucial theorem proved in [24] as a generalization of a result originally appeared in [9].

Theorem 2.20 (Horizontal and vertical derivatives)**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a m.m.s.,* $f, g \in S^2(X)$ *and* π *a plan representing the gradient of* g*.*

Then

$$\begin{aligned} \int \mathrm{D}^- f(\nabla g) \, \mathrm{d}(e_0)_\sharp \pi &\leq \varliminf_{t \downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\pi(\gamma) \\ &\leq \varlimsup_{t \downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\pi(\gamma) \leq \int \mathrm{D}^+ f(\nabla g) \, \mathrm{d}(e_0)_\sharp \pi. \end{aligned} \tag{2.44}$$

Proof. Write inequality (2.42) for the function $g + \varepsilon f$ and subtract inequality (2.43) to get

$$\varlimsup_{t \downarrow 0} \varepsilon \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, \mathrm{d}\pi \leq \frac{1}{2} \int |\mathrm{D}(g + \varepsilon f)|^2 - |\mathrm{D}g|^2 \, \mathrm{d}(e_0)_\sharp \pi$$

Divide by $\varepsilon > 0$ (resp. $\varepsilon < 0$), let $\varepsilon \downarrow 0$ (resp. $\varepsilon \uparrow 0$) and use the dominate convergence theorem to conclude. □

2.4.2 The result

We are now ready to define what it is a solution of the continuity equation (2.36) in a metric measure context.

Definition 2.21 (Solutions of $\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0$)**.** Let $\{\mu_t\}_t \subset \mathcal{P}_2(X)$ be a $\mathcal{W}_2$-continuous curve with bounded compression and $\{\phi_t\}_{t \in [0,1]} \subset S^2(X)$ a given family. We say that $\{\mu_t\}_t$ solves the continuity equation

$$\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0, \tag{2.45}$$

provided:

i) for every $f \in S^2(X)$ the maps $(t,x) \mapsto \mathrm{D}^{\pm} f(\nabla\phi_t)(x)$ are $L \times \mathfrak{m}$ measurable and the map $\tilde{N} : [0,1] \mapsto [0,\infty]$ given by

$$\frac{1}{2}\tilde{N}_t^2 := \operatorname*{ess\,sup}_{f \in S^2(X)} \int \mathrm{D}^+ f(\nabla\phi_t)\,\mathrm{d}\mu_t - \frac{1}{2}\|f\|_{\mu_t}^2 \tag{2.46}$$

is in $L^2(0,1)$ where ess sup has the same meaning as we mentioned before.

ii) for every $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f\,\mathrm{d}\mu_t$ is absolutely continuous and satisfies

$$\int \mathrm{D}^- f(\nabla\phi_t)\,\mathrm{d}\mu_t \le \frac{\mathrm{d}}{\mathrm{d}t}\int f\,\mathrm{d}\mu_t \le \int \mathrm{D}^+ f(\nabla\phi_t)\,\mathrm{d}\mu_t, \qquad a.e.\ t. \tag{2.47}$$

We then have the following result, analogous to the implication $(ii) \Rightarrow (i)$ of Theorem 4.9. As already recalled, even in the smooth framework not all a.c. curves solve (2.36), so the other implication is in general false.

Proposition 2.22. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a m.m.s. and $\{\mu_t\}_t \subset \mathcal{P}_2(X)$ a continuous curve with bounded compression solving the continuity equation (2.45) for some given family $\{\phi_t\}_{t\in[0,1]} \subset S^2(X)$.*

Then $\{\mu_t\}_t$ is 2-a.c. and we have $|\dot{\mu}_t| \le \tilde{N}_t$ for a.e. $t \in [0,1]$.

If furthermore $S^2(X)$ is separable, then $\tilde{N}_t = |\dot{\mu}_t| = \|\phi_t\|_{\mu_t}$ for a.e. t.

Proof. We claim that for every $f \in S^2(X)$ it holds

$$\max\left\{\left|\int \mathrm{D}^+ f(\nabla\phi_t)\,\mathrm{d}\mu_t\right|, \left|\int \mathrm{D}^- f(\nabla\phi_t)\,\mathrm{d}\mu_t\right|\right\} \le \|f\|_{\mu_t}\tilde{N}_t, \qquad a.e.\ t. \tag{2.48}$$

To this aim, fix a representative of $\tilde{N}$, a function $f \in S^2(X)$ and notice that for every $\lambda \ge 0$, by definition of $\tilde{N}$ and the second of (2.40) we have

$$\lambda \int \mathrm{D}^+ f(\nabla\phi_t)\,\mathrm{d}\mu_t \le \frac{\lambda^2}{2}\|f\|_{\mu_t}^2 + \frac{1}{2}\tilde{N}_t^2, \tag{2.49}$$

for $\mathcal{L}^1$-a.e. t. Replacing f with $-f$ and recalling that

$$-\int \mathrm{D}^+ f(\nabla\phi_t)\,\mathrm{d}\mu_t = \int \mathrm{D}^-(-f)(\nabla\phi_t)\,\mathrm{d}\mu_t \le \int \mathrm{D}^+(-f)(\nabla\phi_t)\,\mathrm{d}\mu_t,$$

we deduce that (2.49) holds for every $\lambda \in \mathbb{R}$. In particular, there is a $\mathcal{L}^1$-negligible set $\mathcal{N} \subset [0,1]$ such that for every $t \in [0,1] \setminus \mathcal{N}$ the inequality (2.49) holds for every $\lambda \in \mathbb{Q}$. Given that all the terms in (2.49) are continuous in λ, we deduce that (2.49) holds for every $t \in [0,1] \setminus \mathcal{N}$ and every $\lambda \in \mathbb{R}$, which yields

$$\left|\int \mathrm{D}^+ f(\nabla\phi_t)\,\mathrm{d}\mu_t\right| \le \|f\|_{\mu_t}\tilde{N}_t, \qquad a.e.\ t.$$

Arguing analogously with $\mathrm{D}^-f(\nabla\phi_t)$ in place of $\mathrm{D}^+f(\nabla\phi_t)$ we obtain (2.48).

Now define a linear operator $L : S^2(X) \mapsto L^1(0,1)$ as

$$L(f)(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int f\,\mathrm{d}\mu_t,$$

and observe that for every $t, s \in [0,1]$, $t < s$, taking into account (2.47) and (2.48) we have

$$\begin{aligned}\int_t^s |L(f)(r)|\,\mathrm{d}r &\leq \int_t^s \max\left\{\left|\int \mathrm{D}^+f(\nabla\phi_r)\,\mathrm{d}\mu_r\right|, \left|\int \mathrm{D}^-f(\nabla\phi_r)\,\mathrm{d}\mu_r\right|\right\}\mathrm{d}r\\ &\leq \int_t^s \|f\|_{\mu_r}\tilde{N}_r\,\mathrm{d}r \leq \sqrt{\int_t^s \tilde{N}_r^2\,\mathrm{d}r}\sqrt{\int_t^s \|f\|^2_{\mu_r}\,\mathrm{d}r}.\end{aligned}$$

Hence we can apply first Lemma 2.8 (with $G := \tilde{N}$) and then Theorem 4.9 to deduce that $\{\mu_t\}_t$ is 2-a.c. with $|\dot{\mu}_t| \leq \tilde{N}_t$ for a.e. $t \in [0,1]$.

If $S^2(X)$ is separable, then arguing exactly as in the proof of Proposition 2.13 and using the convexity (resp. concavity) of $f \mapsto \int \mathrm{D}^+f(\nabla\phi_t)\,\mathrm{d}\mu_t$ (resp. $f \mapsto \int \mathrm{D}^-f(\nabla\phi_t)\,\mathrm{d}\mu_t$) expressed in (2.41) we deduce the existence of a $\mathcal{L}^1$-negligible set $\mathcal{N} \subset [0,1]$ such that for $t \in [0,1] \setminus \mathcal{N}$ the conclusions $(i), (ii)$ of Proposition 2.13 hold and furthermore

$$\int \mathrm{D}^-f(\nabla\phi_t)\,\mathrm{d}\mu_t \leq L_t(f) \leq \int \mathrm{D}^+f(\nabla\phi_t)\,\mathrm{d}\mu_t, \qquad \forall f \in S^2(X).$$

Choosing $f := \phi_t$ and recalling (2.39) we obtain

$$\|\phi_t\|^2_{\mu_t} = L_t(\phi_t) \leq \|\phi_t\|_{\mu_t}\|L_t\|^*_{\mu_t}, \qquad \forall t \in [0,1] \setminus \mathcal{N},$$

and hence $\|\phi_t\|_{\mu_t} \leq \|L_t\|^*_{\mu_t} = N_t = |\dot{\mu}_t|$ for a.e. t. On the other hand, letting $(f_n) \subset S^2(X)$ be a countable dense set, by (2.37) we know that $\frac{1}{2}\tilde{N}_t^2 = \sup_n \mathrm{D}^+f_n(\nabla\phi_t) - \frac{1}{2}\|f_n\|^2_{\mu_t}$ for a.e. t and thus $\tilde{N}_t \leq \|\phi_t\|_{\mu_t}$ for a.e. t. □

The continuity equation (2.45) has very general relations with the concept of 'plans representing gradients', as shown by the following result:

Theorem 2.23. *Let $\{\mu_t\}_t \subset \mathcal{P}_2(X)$ be a 2-a.c. curve with bounded compression, $(t,x) \mapsto \phi_t(x)$ a Borel map such that $\phi_t \in S^2(X)$ for a.e. $t \in [0,1]$ and π a lifting of $\{\mu_t\}_t$.*

The following are true.

i) Assume that $(\mathrm{restr}_t^1)_\sharp\pi$ represents the gradient of $(1-t)\phi_t$ for a.e. $t \in [0,1]$. Then $\{\mu_t\}_t$ solves the continuity equation (2.45).

ii) Assume that $S^2(X)$ is separable and that $\{\mu_t\}_t$ solves the continuity equation (2.45). Then $(\mathrm{restr}_t^1)_\sharp\pi$ represents the gradient of $(1-t)\phi_t$ for a.e. $t\in[0,1]$.

Proof.
(i) Let $\mathcal{A}\subset(0,1)$ be the set of t's such that $(\mathrm{restr}_t^1)_\sharp\pi$ represents the gradient of $(1-t)\phi_t$, so that by assumption we know that $\mathcal{L}^1(\mathcal{A})=1$. Pick $f\in S^2(X)$ and recall that by Theorem 4.9 we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f\,\mathrm{d}\mu_t = L_t(f), \qquad a.e.\ t\in[0,1]. \tag{2.50}$$

Fix $t\in\mathcal{A}$ such that (4.7) holds and notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f\,\mathrm{d}\mu_t = \lim_{h\downarrow 0}\int\frac{f(\gamma_{t+h})-f(\gamma_t)}{h}\,\mathrm{d}\pi(\gamma) = \frac{1}{1-t}\lim_{h\downarrow 0}\int\frac{f(\gamma_h)-f(\gamma_0)}{h}\,\mathrm{d}\pi_t(\gamma),$$

so that recalling (2.44) and (2.40) we conclude.

(ii) With exactly the same approximation procedure used in the proof of Proposition 2.13, we see that there exists a $\mathcal{L}^1$-negligible set $\mathcal{N}\subset[0,1]$ such that the thesis of Proposition 2.13 is fulfilled and furthermore for every $t\in[0,1]\setminus\mathcal{N}$ we have

$$\int \mathrm{D}^-f(\nabla\phi_t)\,\mathrm{d}\mu_t \le L_t(f)\le\int\mathrm{D}^+f(\nabla\phi_t)\,\mathrm{d}\mu_t, \qquad \forall f\in S^2(X). \tag{2.51}$$

Fix $t\in[0,1]\setminus\mathcal{N}$ and observe that by point (i) of Proposition 2.13 we have that

$$|\dot\mu_t|^2 = \lim_{h\downarrow 0}\frac{1}{h}\iint_t^{t+h}|\dot\gamma_s|^2\,\mathrm{d}\pi(\gamma) = \frac{1}{(1-t)^2}\lim_{h\downarrow 0}\frac{1}{h}\iint_0^h|\dot\gamma_s|^2\,\mathrm{d}\pi_t(\gamma). \tag{2.52}$$

Now pick $f:=\phi_t$ in (2.51) and recall the identity $D^\pm f(\nabla f)=|\mathrm{D}f|^2$ $\mathfrak{m}$-a.e. valid for every $f\in S^2(X)$ to deduce

$$\int|\mathrm{D}\phi_t|^2\,\mathrm{d}\mu_t = L_t(\phi_t) = \lim_{h\downarrow 0}\int\phi_t\,\mathrm{d}\frac{\mu_{t+h}-\mu_t}{h} = \frac{1}{1-t}\lim_{h\downarrow 0}\int\frac{\phi_t(\gamma_h)-\phi_t(\gamma_0)}{h}\,\mathrm{d}\pi_t(\gamma).$$

This last identity, (2.52) and the fact that $\|L_t\|^*_{\mu_t}=|\dot\mu_t|$ ensure that π_t represents the gradient of $(1-t)\phi_t$, as claimed. □

2.5 Two important examples

We conclude the paper discussing two important examples of absolutely continuous curves on $\mathcal{P}_2(X)$: the heat flow and the geodesics. These examples already appeared in the literature [8, 22, 28], we report them here only to show that they are consistent with the concepts we introduced.

We start with the heat flow. Recall that the map $E : L^2(X,\mathfrak{m}) \mapsto [0,\infty]$ given by

$$E(f) := \begin{cases} \dfrac{1}{2}\displaystyle\int |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m}, & \text{if } f \in S^2(X), \\ +\infty, & \text{otherwise,} \end{cases}$$

is convex, lower semicontinuous and with dense domain. Being $L^2(X)$ an Hilbert space, we then know by the classical theory of gradient flows in Hilbert spaces (see e.g. [5] and references therein) that for any $\rho \in L^2(X)$ there exists a unique continuous curve $[0,\infty) \ni t \mapsto \rho_t \in L^2(X)$ with $f_0 = f$ which is locally absolutely continuous on $(0,\infty)$ and that satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t \in -\partial^- E(\rho_t), \qquad a.e.\ t > 0.$$

As in [8], we shall call any such gradient flow a *heat flow*. It is immediate to check that defining $D(\Delta) := \{\rho : \partial^- E(\rho) \neq \emptyset\}$ and for $\rho \in D(\Delta)$ the Laplacian $\Delta\rho$ as the opposite of the element of minimal norm in $\partial^- E(\rho)$, for any heat flow (ρ_t) we have $\rho_t \in D(\Delta)$ for any $t > 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = \Delta\rho_t, \qquad a.e.\ t > 0,$$

in accordance with the classical case. It is our aim now to check that, under reasonable assumptions, putting $\mu_t := \rho_t\mathfrak{m}$, the curve $\{\mu_t\}_t$ solves

$$\partial_t\mu_t + \nabla\cdot(\nabla(-\log\rho_t)\mu_t) = 0.$$

To this aim, recall that for any heat flow (ρ_t) we have the weak maximum principle

$$\rho_0 \le C \text{ (resp. } \rho_0 \ge c) \ \mathfrak{m}-a.e. \qquad \Rightarrow \qquad \rho_t \le C \text{ (resp. } \rho_t \ge c) \ \mathfrak{m}-a.e. \text{ for any } t > 0, \tag{2.53}$$

where $c, C \in \mathbb{R}$ and the estimate

$$\int_0^\infty \|\Delta\rho_t\|^2_{L^2(X)} \,\mathrm{d}t \le E(\rho_t). \tag{2.54}$$

Furthermore, if $\mathfrak{m} \in \mathcal{P}(X)$ then $L^2(X,\mathfrak{m}) \subset L^1(X,\mathfrak{m})$ and the mass preservation property holds:

$$\int \rho_t \,\mathrm{d}\mathfrak{m} = \int \rho_0 \,\mathrm{d}\mathfrak{m}, \qquad \forall t > 0. \tag{2.55}$$

See [8] for the simple proof of these facts.

We can now state our result concerning the heat flow as solution of the continuity equation.

Proposition 2.24 (Heat flow)**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be a metric measure space with $\mathfrak{m} \in \mathcal{P}_2(X)$ and ρ_0 a probability density such that $c \le \rho_0 \le C$ $\mathfrak{m}$-a.e. for some $c, C > 0$ (in particular $\rho_0 \in L^2(X,\mathfrak{m})$) and $E(\rho_0) < \infty$. Let (ρ_t) be the heat flow starting from from ρ_0.*

Then the curve $[0,1] \ni t \mapsto \mu_t := \rho_t \mathfrak{m}$ *solves the continuity equation*

$$\partial_t \mu_t + \nabla \cdot (\nabla(-\log \rho_t)\mu_t) = 0.$$

Proof. By (2.55) and (2.53) we know that ρ_t is a probability density for every t and (2.53) again and the assumptions $\mathfrak{m} \in \mathcal{P}_2(X)$ and $\rho_0 \le C\mathfrak{m}$ ensure that $\mu_t \in \mathcal{P}_2(X)$ for every $t \in [0,1]$ and that $\{\mu_t\}_t$ has bounded compression. The $\mathcal{W}_2$-continuity of $(\{\mu_t\}_t$ is a simple consequence of the L^2-continuity of (ρ_t) and the bounds $\rho_t \le C$, $\mathfrak{m} \in \mathcal{P}_2(X)$. Also, recalling the chain rule (2.18) and the maximum principle (2.53) we have

$$\|\log \rho_t\|_{\mu_t}^2 = \int |\mathrm{D}(\log \rho_t)|^2 \,\mathrm{d}\mu_t = \int \frac{|\mathrm{D}\rho_t|^2}{\rho_t} \,\mathrm{d}\mathfrak{m} \le \frac{1}{c} \int |\mathrm{D}\rho_t|^2 \,\mathrm{d}\mathfrak{m},$$

so that (2.54) ensures that $\int_0^1 \|\log \rho_t\|_{\mu_t}^2 < \infty$, which directly yields that point (i) of Definition 2.21 is fulfilled.

It remains to prove that for every $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f \,\mathrm{d}\mu_t$ is absolutely continuous and fulfills

$$\int \mathrm{D}^- f(\nabla(-\log \rho_t)) \,\mathrm{d}\mu_t \le \frac{\mathrm{d}}{\mathrm{d}t} \int f \,\mathrm{d}\mu_t \le \int \mathrm{D}^+ f(\nabla(-\log \rho_t)) \,\mathrm{d}\mu_t.$$

Taking into account the chain rule (2.40) the above can be written as

$$-\int \mathrm{D}^+ f(\nabla \rho_t) \,\mathrm{d}\mathfrak{m} \le \frac{\mathrm{d}}{\mathrm{d}t} \int f \,\mathrm{d}\mu_t \le -\int \mathrm{D}^- f(\nabla \rho_t) \,\mathrm{d}\mathfrak{m}. \tag{2.56}$$

Pick $f \in L^2 \cap S^2(X)$ and notice that $t \mapsto \int f \,\mathrm{d}\mu_t = \int f\rho_t \,\mathrm{d}\mathfrak{m}$ is continuous on $[0,1]$ and locally absolutely continuous on $(0,1]$. The inequality

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \int f\rho_t \,\mathrm{d}\mathfrak{m} \right| \le \int |f||\Delta\rho_t| \,\mathrm{d}\mathfrak{m} \le \frac{1}{2}\|f\|_{L^2}^2 + \frac{1}{2}\|\Delta\rho_t\|_{L^2}^2,$$

the bound (2.54) and the assumption $E(\rho_0) < \infty$ ensure that the derivative of $t \mapsto \int f\rho_t \,\mathrm{d}\mathfrak{m}$ is in $L^1(0,1)$, so that this function is absolutely continuous on $[0,1]$.

The fact that $-\frac{\mathrm{d}}{\mathrm{d}t}\rho_t \in -\partial^- E(\rho_t)$ for a.e. t grants that for $\varepsilon \in \mathbb{R}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varepsilon f\rho_t \,\mathrm{d}\mathfrak{m} = \int \varepsilon f \frac{\mathrm{d}}{\mathrm{d}t}\rho_t \,\mathrm{d}\mathfrak{m} \le E(\rho_t - \varepsilon f) - E(f), \qquad \text{a.e. } t. \tag{2.57}$$

Divide by $\varepsilon > 0$ and let $\varepsilon \downarrow 0$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f\rho_t \,\mathrm{d}\mathfrak{m} \le \varliminf_{\varepsilon\downarrow 0} \frac{E(\rho_t - \varepsilon f) - E(f)}{\varepsilon} = \lim_{\varepsilon\downarrow 0} \int \frac{|\mathrm{D}(\rho_t - \varepsilon f)|^2 - |\mathrm{D}\rho_t|^2}{2\varepsilon} \,\mathrm{d}\mathfrak{m} = -\int \mathrm{D}^- f(\nabla\rho_t) \,\mathrm{d}\mathfrak{m},$$

which is the second inequality in (3.7). The first one is obtained starting from (2.57), dividing by $\varepsilon < 0$ and letting $\varepsilon \uparrow 0$.

The general case of $f \in L^1 \cap S^2(X)$ can now be obtained with a simple truncation argument, we omit the details. $\square$

We now turn to the study of geodesics. In the smooth Euclidean/Riemannian framework a geodesic $\{\mu_t\}_t$ solves

$$\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0,$$

where $\phi_t := Q_{1-t}(-\varphi^c)$ and φ is a Kantorovich potential inducing $\{\mu_t\}_t$.

We want to show that the same holds on metric measure spaces, at least for geodesics with bounded compressions. This will be achieved as a simple consequence of Theorem 2.23 and the following fact:

Theorem 2.25. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a m.m.s., $\{\mu_t\}_t \subset \mathcal{P}_2(X)$ a geodesic with bounded compression and $\varphi \in S^2(X)$ a Kantorovich potential from μ_0 to μ_1. Then:*

i) Any lifting π of $\{\mu_t\}_t$ represents the gradient of $-\varphi$.

ii) For any $t \in (0, 1]$ the function $(1-t)Q_{1-t}(-\varphi^c)$ is a Kantorovich potential from μ_t to μ_1.

Point (i) of this theorem is a restatement of the metric Brenier theorem proved in [8], while point (ii) is a general fact about optimal transport in metric spaces whose proof can be found in [43] or [3].

In stating the continuity equation for geodesics we shall make use of the fact that for $\mu, \nu \in \mathcal{P}_2(X)$ with bounded support, there always exists a Kantorovich potential from μ to ν which is constant outside a bounded set: it is sufficient to pick any Kantorovich potential satisfying (2.34) and proceed with a truncation argument. This procedure ensures that if $\mathfrak{m}$ gives finite mass to bounded sets, then these Kantorovich potentials are in $S^2(X)$.

We then have the following result:

Proposition 2.26 (Geodesics)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a m.m.s. with $\mathfrak{m}$ giving finite mass to bounded sets, $\{\mu_t\}_t \subset \mathcal{P}_2(X)$ a geodesic with bounded compression such that μ_0, μ_1 have bounded supports and φ a Kantorovich potential from μ_0 to μ_1 it which is constant outside a bounded set.*

Then

$$\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0,$$

where $\phi_t := -Q_{1-t}(-\varphi^c)$ *for every* $t \in [0,1]$.

Proof. The assumption that φ is constant outside a bounded set easily yields that φ^c is Lipschitz and constant outside a bounded set and that for some $B \subset X$ bounded, ϕ_t is constant outside B for any $t \in [0,1]$. Also, recalling point (i) of Proposition 2.6 we get that the ϕ_t's are uniformly Lipschitz so that the assumption that $\mathfrak{m}$ gives finite mass to bounded sets yields that $\sup_{t\in[0,1]} \|\phi_t\|_{S^2} < \infty$. In particular, the function $\tilde{N}$ defined in (2.46) is bounded and hence in $L^2(0,1)$, so that the statement makes sense.

Now let π be a lifting of $\{\mu_t\}_t$ and notice that π is a test plan so that $t \mapsto \int f\, \mu_t$ is absolutely continuous. For $t \in [0,1)$ the plan $\pi_t := (\mathrm{restr}_t^1)_\sharp \pi$ is a lifting of $s \mapsto \mu_{t+s(1-t)}$. Thus by Theorem 2.25 above we deduce that π_t represents the gradient of $(1-t)\phi_t$.

The conclusion follows by point (i) of Theorem 2.23. □

In many circumstances, both heat flows and geodesics have regularity which go slightly beyond that of absolute continuity. Let us propose the following definition:

Definition 2.27 (Weakly C^1 curves)**.** Let $\{\mu_t\}_t \subset \mathcal{P}_2(X)$ be a 2-a.c. curve with bounded compression. We say that $\{\mu_t\}_t$ is weakly C^1 provided for any $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f\, \mathrm{d}\mu_t$ is C^1.

In presence of weak C^1 regularity, the description of the operators L_t in Theorem 4.9 can be simplified avoiding the use of the technical Lemma 2.8: it is sufficient to define $L_t : S^2(X) \mapsto \mathbb{R}$ by

$$L_t(f) := \frac{\mathrm{d}}{\mathrm{d}t} \int f\, \mathrm{d}\mu_t, \qquad \forall f \in S^2(X).$$

Let us now discuss some cases where the heat flow and the geodesics are weakly C^1. We recall that $(X, \mathrm{d}, \mathfrak{m})$ is said *infinitesimally strictly convex* provided

$$\mathrm{D}^- f(\nabla g) = \mathrm{D}^+ f(\nabla g), \qquad \mathfrak{m}-\text{ a.e. } \forall f, g \in S^2(X).$$

We then have the following regularity result:

Proposition 2.28 (Weak C^1 regularity for the heat flow)**.** *With the same assumptions of Proposition 2.24, assume furthermore that* $(X, \mathrm{d}, \mathfrak{m})$ *is infinitesimally strictly convex.*

Then $\{\mu_t\}_t$ *is weakly* C^1.

Proof. We have already seen in the proof of Proposition 2.24 that for any $\rho \in D(\Delta) = D(\partial^- E)$ and $v \in -\partial^- E(\rho) \subset L^2(X, \mathfrak{m})$ we have

$$\int \mathrm{D}^- f(\nabla \rho)\, \mathrm{d}\mathfrak{m} \leq \int f v \, \mathrm{d}\mathfrak{m} \leq \int \mathrm{D}^+ f(\nabla \rho)\, \mathrm{d}\mathfrak{m}.$$

Thus if $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally strictly convex, the set $\partial^- E(\rho)$ contains at most one element. The conclusion then follows from the weak-strong closure of $\partial^- E$. □

We now turn to geodesics: we will discuss only the case of *infinitesimally Hilbertian* spaces, although weak C^1 regularity is valid on more general circumstances (see [22]). We recall that $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally Hilbertian provided

$$\|f + g\|^2_{S^2} + \|f - g\|^2_{S^2} = 2\|f\|^2_{S^2} + 2\|g\|^2_{S^2}, \qquad \forall f, g \in S^2(X),$$

and that on infinitesimally Hilbertian spaces we have

$$\mathrm{D}^- f(\nabla g) = \mathrm{D}^+ f(\nabla g) = \mathrm{D}^- g(\nabla f) = \mathrm{D}^+ g(\nabla f), \qquad \mathfrak{m} - \text{ a.e. } \forall f, g \in S^2(X),$$

so that in particular infinitesimally Hilbertian spaces are infinitesimally strictly convex. The common value of the above expressions will be denoted by $\nabla f \cdot \nabla g$.

The proof of weak C^1 regularity is based on the following lemma, proved in [22]:

Lemma 2.29 ('Weak-strong' convergence)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be an infinitesimally Hilbert space. Also:*

i) Let $\{\mu_n\} \subset \mathcal{P}_2(X)$ a sequence with uniformly bounded densities, such that letting ρ_n be the density of μ_n we have and $\rho_n \to \rho$ $\mathfrak{m}$-a.e. for some probability density ρ. Put $\mu := \rho\mathfrak{m}$.

ii) Let $\{f_n\} \subset S^2(X)$ be such that:

$$\sup_{n\in\mathbb{N}} \int |\mathrm{D} f_n|^2 \, \mathrm{d}\mathfrak{m} < \infty,$$

and assume that $f_n \to f$ $\mathfrak{m}$-a.e. for some Borel function f.

iii) Let $(g_n) \subset S^2(X)$ and $g \in S^2(X)$ such that $g_n \to g$ $\mathfrak{m}$-a.e. as $n \to +\infty$ and:

$$\sup_{n\in\mathbb{N}} \int |\mathrm{D} g_n|^2 \, \mathrm{d}\mathfrak{m} < \infty, \qquad \lim_{n\to\infty} \int |\mathrm{D} g_n|^2 \, \mathrm{d}\mu_n = \int |\mathrm{D} g|^2 \, \mathrm{d}\mu.$$

Then

$$\lim_{n\to\infty} \int \nabla f_n \cdot \nabla g_n \, d\mu_n = \int \nabla f \cdot \nabla g \, d\mu.$$

Proposition 2.30 (Weak C^1 regularity for geodesics)**.** *With the same assumptions of Proposition 2.26 assume furthermore that $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally Hilbertian and that for the densities ρ_t of μ_t we have that $\rho_s \to \rho_t$ in $L^1(X, \mathfrak{m})$ as $s \to t$.*

Then $\{\mu_t\}_t$ is a weakly C^1 curve.

Proof. By Proposition 2.26, its proof and taking into account the assumption of infinitesimal Hilbertianity we know that for every $t \in [0, 1)$ and $f \in L^1 \cap S^2(X)$ we have

$$\lim_{h \downarrow 0} \frac{\int f \,\mathrm{d}\mu_{t+h} - \int f \,\mathrm{d}\mu_t}{h} = \int \nabla f \cdot \nabla \phi_t \,\mathrm{d}\mu_t. \tag{2.58}$$

To conclude it is enough to show that the right hand side of the above expression is continuous in t. Pick $t \in [0, 1)$ and let $\{t_n\}_n \subset [0, 1]$ be a sequence converging to t. Up to pass to a subsequence, not relabeled, and using the assumption of strong convergence in $L^1(X, \mathfrak{m})$ of ρ_{t_n} to ρ_t, we can assume that $\rho_{t_n} \to \rho_t$ $\mathfrak{m}$-a.e. as $n \to \infty$. The proof of Proposition 2.26 grants that $\sup_n \|\phi_{t_n}\|_{S^2} < \infty$ and that by point (ii) of Proposition 2.6 we know that $\phi_{t_n}(x) \to \phi_t(x)$ as $n \to \infty$ for every $x \in X$. Finally, it is obvious that $\lim_{n\to\infty} \int |\mathrm{D}f|^2 \,\mathrm{d}\mu_{t_n} = \int |\mathrm{D}f|^2 \,\mathrm{d}\mu_t$ (because weak convergence in duality with $C_b(X)$ plus uniform bound on the density grant weak convergence in duality with $L^1(X, \mathfrak{m})$).

Thus by Lemma 2.29 we deduce the desired continuity of the right hand side of (2.58) for $t \in [0, 1)$. Continuity at $t = 1$ is obtained by considering the geodesic $t \mapsto \mu_{1-t}$. □

It is worth recalling that the assumptions of Proposition 2.30 are fulfilled on $\mathrm{RCD}(K, \infty)$ spaces when $\{\mu_t\}_t$ is a (in fact ‘the’) geodesic connecting two measures with bounded support and bounded density (see [22]).

Chapter 3

Sobolev Spaces on Warped Products

Abstract
In this chapter, we study the structure of Sobolev spaces on the cartesian/warped products of a given metric measure space and an interval. We prove the 'Pythagoras theorem' for both cartesian products and warped products, and prove Sobolev-to-Lipschitz property for warped products under a certain curvature-dimension condition.

Résumé
Dans ce chapitre, nous étudions les espaces de Sobolev sur le produit tordu de l'ensemble des réels et d'un espace métrique mesuré. Nous montrons le 'théorème de Pythagore' pour les produits cartésiens et des produits tordus, sans condition de courbure-dimension. En suite, nous montrons la propriété Sobolev-à-Lipschitz sous une certaine condition de courbure-dimension.

The results in this chapter are contained in [27].

3.1 Introduction

There is a well established definition of the space $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ of real valued Sobolev functions defined on a metric measure space $(X, \mathrm{d}, \mathfrak{m})$, see e.g. [41] for an overview of the topic and [1] for more recent developments. A function $f \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ comes with a function $|\mathrm{D}f|_X \in L^2(X, \mathfrak{m})$, called minimal weak upper gradient, playing the role of what the modulus of the distributional differential is in the smooth setting.

In this paper we are interested in the structure of the Sobolev spaces and the correspoding minimal weak upper gradients under some basic geometric constructions. The basic problem is the following. Let $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ be two metric measure spaces. We consider the space $X \times Y$ endowed with the product measure $\mathfrak{m}_c := \mathfrak{m}_X \times \mathfrak{m}_Y$ and the product distance d_c defined as

$$\mathrm{d}_c^2\big((x_1, y_1), (x_2, y_2)\big) := \mathrm{d}_X^2(x_1, x_2) + \mathrm{d}_Y^2(y_1, y_2), \qquad \forall x_1, x_2 \in X,\ y_1, y_2 \in Y.$$

Then one asks what is the relation between Sobolev functions on $X \times Y$ and those on X, Y. Guided by the Euclidean case, one might conjecture that $f \in W^{1,2}(X \times Y)$ if and only for $\mathfrak{m}_X$-a.e. x the function $y \mapsto f(x, y)$ is in $W^{1,2}(Y)$, for $\mathfrak{m}_Y$-a.e. y the function $x \mapsto f(x, y)$ is in $W^{1,2}(X)$ and the quantity

$$\sqrt{|\mathrm{D}f(\cdot, y)|_X^2(x) + |\mathrm{D}f(x, \cdot)|_Y^2(y)}$$

is in $L^2(X \times Y, \mathfrak{m}_c)$. Then one expects the above quantity to coincide with $|\mathrm{D}f|_{X\times Y}$.

Curiously, this kind of problem has not been studied until recently and, despite the innocent-looking statement, the full answer is not yet known.

The first result in this direction has been obtained in [9], where it has been proved that the conjecture is true under the very restrictive assumption that the spaces considered satisfy the, there introduced, $\mathrm{RCD}(K, \infty)$ condition for some $K \in \mathbb{R}$. Such restriction was necessary to use some regularization property of the heat flow.

The curvature condition has been dropped in the more recent paper [10]. There the authors prove that the above conjecture holds provided either both the base spaces are doubling and support a weak local 1-2 Poincaré inequality, or on both the spaces the integral of the local Lipschitz constant squared is a quadratic form on the space of Lipschitz functions.

Our contribution to the topic is the proof that the above conjecture is always true, provided one of the two spaces is $\mathbb{R}$ or a closed subinterval of $\mathbb{R}$. Our strategy is new and also allows to cover the case of warped product of a space and a closed interval, thus permitting to consider basic geometric constructions like that of cone and spherical subspension of a given space.

In fact, this line of research is motivated by the study of geometric properties of metric measure spaces, typically having Ricci curvature bounded from below in the appropriate weak sense, via the study of Sobolev functions on them (see in particular [22] and [31] for two examples where this project has been carried out).

In the last section of the paper we study the Sobolev-to-Lipschitz property (see Section 3.3.3 for the definition) of a warped product. Such notion, introduced in [22], is key to deduce precise metric information from the study of Sobolev functions. It is therefore interesting to ask whether warped products have this property. We will show that this is the case under very general assumptions on the warping function, assuming that the base space X is $\mathrm{RCD}(K, \infty)$ and doubling.

3.2 Preliminaries

3.2.1 Metric measure spaces

Let (X, d) be a complete metric space. By a curve γ we shall typically denote a a continuous map $\gamma : [0, 1] \mapsto X$, although sometimes curves defined on different intervals will be considered. The space of curves on $[0, 1]$ with values in X is denoted by $C([0, 1], X)$. The space $C([0, 1], X)$ equipped with the uniform norm is a complete metric space.

We define the length of γ by

$$l[\gamma] := \sup_{\tau} \sum_{i=1}^{n} \mathrm{d}(\gamma(t_{i-1}), \gamma(t_i))$$

where $\tau := \{0 = t_0, t_1, ..., t_n = 1\}$ is a partition of $[0, 1]$. The supreme here can be changed to 'lim' and the limit is taken with respect to the refinement ordering of partitions.

The space (X, d) is said to be a length space if for any $x, y \in X$ we have

$$\mathrm{d}(x, y) = \inf_{\gamma} l[\gamma]$$

where the infimum is taken among all $\gamma \in C([0, 1], X)$ which connect x and y.

If the infimum is always a minimum, then the space is called geodesic space and we call the minimizers pre-geodesics. A geodesic from x to y is any pre-geodesic which is parameterized by constant speed. Equivalently, a geodesic from x to y is a curve γ such that:

$$\mathrm{d}(\gamma_s, \gamma_t) = |s - t|\mathrm{d}(\gamma_0, \gamma_1), \qquad \forall t, s \in [0, 1], \quad \gamma_0 = x, \gamma_1 = y.$$

The space of all geodesics on X will be denoted by Geo(X). It is a closed subset of $C([0, 1], X)$.

Given $p \in [1, +\infty]$ and a curve γ, we say that γ belongs to $AC^p([0,1], X)$ if

$$\mathrm{d}(\gamma_s, \gamma_t) \leq \int_s^t G(r)\,\mathrm{d}r, \qquad \forall\ t, s \in [0,1],\ s < t$$

for some $G \in L^p([0,1])$. In particular, the case $p = 1$ corresponds to absolutely continuous curves, whose class is denoted by $AC([0,1], X)$. It is known (see for instance Theorem 1.1.2 of [5]) that for $\gamma \in AC([0,1], X)$, there exists an a.e. minimal function G satisfying this inequality, called the metric derivative which can be computed for a.e. $t \in [0,1]$ as

$$|\dot{\gamma}_t| := \lim_{h \to 0} \frac{\mathrm{d}(\gamma_{t+h}, \gamma_t)}{|h|}.$$

It is known that (see for example [11], [19]) the length of a curve $\gamma \in AC([0,1], X)$ can be computed as

$$l[\gamma] := \int_0^1 |\dot{\gamma}_t|\,\mathrm{d}t.$$

In particular, on a length space X we have

$$\mathrm{d}(x, y) = \inf_{\gamma} \int_0^1 |\dot{\gamma}_t|\,\mathrm{d}t$$

where the infimum is taken among all $\gamma \in AC([0,1], X)$ which connect x and y.

Given $f : X \mapsto \mathbb{R}$, the local Lipschitz constant $\mathrm{lip}(f) : X \mapsto [0, \infty]$ is defined as

$$|\mathrm{lip}(f)|(x) := \overline{\lim_{y \to x}} \frac{|f(y) - f(x)|}{\mathrm{d}(x, y)}$$

if x is not isolated, 0 otherwise, while the (global) Lipschitz constant is defined as

$$\mathrm{Lip}(f) := \sup_{x \neq y} \frac{|f(y) - f(x)|}{\mathrm{d}(x, y)}.$$

If (X, d) is a length space, we have $\mathrm{Lip}(f) = \sup_x \mathrm{lip}(f)(x)$.

We are not only interested in metric structure, but also in the interaction between metric and measure. For the metric measure space $(X, \mathrm{d}, \mathfrak{m})$, basic assumptions used in this paper are:

Assumption 3.1. *The metric measure spaces* $(X, \mathrm{d}, \mathfrak{m})$ *satisfies:*

- (X, d) *is a complete and separable length space,*
- $\mathfrak{m}$ *is a non-negative Borel measure with respect to* d *finite on bounded sets,*
- $\mathrm{supp}\,\mathfrak{m} = X$.

Moreover, for brevity we will not distinguish $X, (X, \mathrm{d})$ or $(X, \mathrm{d}, \mathfrak{m})$ when no ambiguities exist. For example, we write $S^2(X)$ instead of $S^2(X, \mathrm{d}\mathfrak{m})$ (see the next section).

3.2.2 Optimal transport and Sobolev functions

The set of Borel probability measures on (X, d) will be denoted by $\mathcal{P}(X)$. We also use $\mathcal{P}_2(X) \subseteq P(X)$ to denote the set of measures with finite 2-moment, i.e. $\mu \in \mathcal{P}_2(X)$ if $\mu \in \mathcal{P}(X)$ and $\int \mathrm{d}^2(x, x_0)\,\mathrm{d}\mu(x) < +\infty$ for some (and thus every) $x_0 \in X$.

For $t \in [0,1]$, the evaluation map $e_t : C([0,1], X) \to X$ is given by

$$e_t(\gamma) := \gamma_t, \qquad \forall \gamma \in C([0,1], X).$$

Then we consider $(\mathcal{P}_2(X), \mathcal{W}_2)$, where we endow $\mathcal{P}_2(X)$ with the 2-Wasserstein distance $\mathcal{W}_2$ defined as:

$$\mathcal{W}_2^2(\nu, \mu) := \inf_{\pi} \int \mathrm{d}^2(x, y)\,\mathrm{d}\pi(x, y),$$

where the inf is taken among all plans π with marginal μ and ν, i.e. $(\Pi_1)_\sharp \pi = \mu$ and $(\Pi_2)_\sharp \pi = \nu$ where $(\Pi_i)_\sharp \pi$ means the measure push forward by the projection maps.

It is know that there exist an optimal transport plan π realizing the infimum in the Kantorovich problem. We denote the set of optimal transport plans between μ and ν by $\mathrm{Opt}(\mu, \nu)$.

Some other important properties of the distance $\mathcal{W}_2$ are the following.

Proposition 3.2 (Geodesics in the Wasserstein space, [3])**.** *Let (X, d) be a metric space and $\mu, \nu \in \mathcal{P}_2(X)$. Then the curve (μ_t) is a constant speed geodesic connecting μ and ν, i.e. it satisfies*

$$\mathcal{W}_2(\mu_s, \mu_t) = |s - t|\mathcal{W}_2(\mu_0, \mu_1), \qquad \forall\ s, t \in [0,1] \tag{3.1}$$

and $\mu_0 = \mu, \mu_1 = \nu$, if and only if there exists a plan $\pi \in \mathcal{P}_2(\mathrm{Geo(X)}) \subseteq \mathcal{P}_2(\mathrm{C([0,1],X)})$ such that

$$\mu_t = (e_t)_\sharp \pi \quad \forall\ t \in [0,1]; \quad (e_0)_\sharp \pi = \mu,\ (e_1)_\sharp \pi = \nu,$$

and $(e_0, e_1)_\sharp \pi \in \mathrm{Opt}(\mu_0, \mu_1)$. We denote the set of these measures in $\mathcal{P}_2(\mathrm{Geo(X)})$ by $\mathrm{OptGeo}(\mu_0, \mu_1)$.

In particular, if X is geodesic, the space $(\mathcal{P}_2(X), \mathcal{W}_2)$ is also a geodesic space.

Moreover, absolutely continuous curves in $(\mathcal{P}_2, \mathcal{W}_2)$ are characterized by the following theorem:

Theorem 3.3 (Superposition principle, [33])**.** *Let (X, d) be a complete and separable metric space, and $(\mu_t) \in AC^2([0,1], \mathcal{P}_2(X))$. Then there exists a measure $\pi \in \mathcal{P}(C([0,1], X))$ concentrated on $AC^2([0,1], X)$ such that:*

$$\begin{aligned} (e_t)_\sharp \pi &= \mu_t, \qquad \forall t \in [0,1] \\ \int |\dot\gamma_t|^2 \,\mathrm{d}\pi(\gamma) &= |\dot\mu_t|^2, \qquad \text{a.e. } t. \end{aligned}$$

Moreover, the minimum of the energy $\int_0^1 \int |\dot\gamma_t|^2 \,\mathrm{d}\pi(\gamma)\,\mathrm{d}t$ among all the plans π' satisfying $(e_t)_\sharp \pi' = \mu_t$ for every $t \in [0,1]$ is obtained by this plan π.

Definition 3.4 (Test plan)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space and $\pi \in \mathcal{P}(C([0,1], X))$. We say that π has bounded compression provided there exists $C > 0$ such that

$$(e_t)_\sharp \pi \le C\mathfrak{m}, \quad \forall t \in [0,1].$$

Then we say that π is a test plan if it has bounded compression, is concentrated on $AC^2([0,1], X)$ and

$$\int_0^1 \int |\dot\gamma_t|^2 \,\mathrm{d}\pi(\gamma)\,\mathrm{d}t < +\infty.$$

The notion of Sobolev function is given by duality with that of test plan:

Definition 3.5 (Sobolev class)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space. A Borel function $f : X \to \mathbb{R}$ belongs to the Sobolev class $S^2(X, \mathrm{d}, \mathfrak{m})$ (resp. $S^2_{loc}(X, \mathrm{d}, \mathfrak{m})$) provided there exists a non-negative function $G \in L^2(X, \mathfrak{m})$ (resp. $L^2_{loc}(X, \mathfrak{m})$) such that

$$\int |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\pi(\gamma) \le \int \int_0^1 G(\gamma_s)|\dot\gamma_s| \,\mathrm{d}s\,\mathrm{d}\pi(\gamma), \quad \forall \text{ test plan } \pi.$$

In this case, G is called a 2-weak upper gradient of f, or simply weak upper gradient.

It is known, see e.g. [8], that there exists a minimal function G in the $\mathfrak{m}$-a.e. sense among all the weak upper gradients of f. We denote such minimal function by $|\mathrm{D}f|$ or $|\mathrm{D}f|_X$ to emphasize which space we are considering and call it minimal weak upper gradient. Notice that if f is Lipschitz, then $|\mathrm{D}f| \le \mathrm{lip}(f)$ $\mathfrak{m}$-a.e., because $\mathrm{lip}(f)$ is a weak upper gradient of f.

It is known that the locality holds for $|\mathrm{D}f|$, i.e. $|\mathrm{D}f| = |\mathrm{D}g|$ a.e. on the set $\{f = g\}$, moreover $S^2_{loc}(X, \mathrm{d}, \mathfrak{m})$ is a vector space and the inequality

$$|\mathrm{D}(\alpha f + \beta g)| \le |\alpha||\mathrm{D}f| + |\beta||\mathrm{D}g|, \qquad \mathfrak{m}-\text{ a.e.}, \tag{3.2}$$

holds for every $f, g \in S^2_{loc}(X, \mathrm{d}, \mathfrak{m})$ and $\alpha, \beta \in \mathbb{R}$ and the space $S^2_{loc} \cap L^\infty_{loc}(X, \mathrm{d}, \mathfrak{m})$ is an algebra, with the inequality

$$|\mathrm{D}(fg)| \leq |f||\mathrm{D}g| + |g||\mathrm{D}f|, \qquad \mathfrak{m}- \text{ a.e.}, \tag{3.3}$$

being valid for any $f, g \in S^2_{loc} \cap L^\infty_{loc}(X, \mathrm{d}, \mathfrak{m})$.

Another basic - and easy to check - property of minimal weak upper gradients that we shall frequently use is their semicontinuity in the following sense: if $(f_n) \subset S^2(X, \mathrm{d}, \mathfrak{m})$ is a sequence $\mathfrak{m}$-a.e. converging to some f and such that $(|\mathrm{D}f_n|)$ is bounded in $L^2(X, \mathfrak{m})$, then $f \in S^2(X, \mathrm{d}, \mathfrak{m})$ and

$$|\mathrm{D}f| \leq G, \qquad \mathfrak{m}- \text{ a.e.},$$

for every L^2-weak limit G of some subsequence of $(|\mathrm{D}f_n|)$.

Then the Sobolev space $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ is defined as $W^{1,2}(X, \mathrm{d}, \mathfrak{m}) := S^2(X, \mathrm{d}, \mathfrak{m}) \cap L^2(X, \mathfrak{m})$ and is endowed with the norm

$$\|f\|^2_{W^{1,2}(X,\mathrm{d},\mathfrak{m})} := \|f\|^2_{L^2(X,\mathfrak{m})} + \||\mathrm{D}f|\|^2_{L^2(X,\mathfrak{m})}.$$

$W^{1,2}(X)$ is always a Banach space, but in general it is not an Hilbert space. Following [24], we say that $(X, \mathrm{d}, \mathfrak{m})$ is an infinitesimally Hilbertian space if $W^{1,2}(X)$ is an Hilbert space.

In [6, 8] the following result has been proved.

Proposition 3.6 (Density in energy of Lipschitz functions)**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a metric measure space and* $f \in W^{1,2}(X)$*. Then there exists a sequence* (f_n) *of Lipschitz functions* L^2*-converging to* f *such that the sequence* $(\mathrm{lip}(f_n))$ L^2*-converges to* $|\mathrm{D}f|$*.*

3.2.3 Product spaces

In this subsection we recall the basic concepts and results about the Cartesian product and the warped product of two spaces. Both metric and metric measure structures are considered.

Given two metric measure spaces $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$, we define their (Cartesian) product as:

Definition 3.7 (Cartesian product)**.** We define the space $(Y \times X, \mathrm{d}_c, \mathfrak{m}_c)$ as the product space $Y \times X$ equipped with the distance $\mathrm{d}_c := \mathrm{d}_Y \times \mathrm{d}_X$ and the measure $\mathfrak{m}_c := \mathfrak{m}_Y \times \mathfrak{m}_X$.

Here $\mathrm{d}_c = \mathrm{d}_Y \times \mathrm{d}_X$ means:

$$\mathrm{d}_c((x_1, y_1), (x_2, y_2)) = \sqrt{\mathrm{d}_Y^2(y_1, y_2) + \mathrm{d}_X^2(x_1, x_2)},$$

for any pairs $(y_1, x_1), (y_2, x_2) \in Y \times X$.

There is a natural and simple to prove (see e.g. [22]) link between Sobolev functions on the product depending on just one variable and Sobolev functions on the base spaces:

Proposition 3.8. *Let $(X, \mathrm{d}_X, \mathfrak{m}_X)$ and $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ be two metric measure spaces, $g \in L^2_{loc}(X)$ and define $f \in L^2_{loc}(Y \times X)$ as $f(y, x) := g(x)$.*

Then $f \in S^2_{loc}(Y \times X)$ if and only if $g \in S^2_{loc}(X)$ and in this case the identity

$$|\mathrm{D}f|_{Y \times X}(y, x) = |\mathrm{D}g|_X(x),$$

holds for $\mathfrak{m}_c$-a.e. (y, x).

To define the warped product metric we need first to introduce the corresponding notion of length:

Definition 3.9 (Warped length of curves)**.** Let (X, d_X) and (Y, d_Y) be two complete and separable metric spaces and $w_\mathrm{d} : Y \to \mathbb{R}^+$ a continuous function. Let $\gamma = (\gamma^Y, \gamma^X)$ be a curve such that γ^X, γ^Y are absolutely continuous. Then the w_d-length of γ is defined as

$$l_w[\gamma] := \lim_\tau \sum_{i=1}^n \sqrt{\mathrm{d}_Y^2(\gamma^Y_{t_{i-1}}, \gamma^Y_{t_i}) + w^2(\gamma^Y_{t_{i-1}})\mathrm{d}_X^2(\gamma^X_{t_{i-1}}, \gamma^X_{t_i})},$$

where $\tau := \{0 = t_0, t_1, ..., t_n = 1\}$ is a partition of $I = [0, 1]$ and the limit is taken with respect to the refinement ordering of partitions.

It is not hard to check that the limit exists and that the formula

$$l_w[\gamma] = \int_0^1 \sqrt{|\dot\gamma^Y_t|^2 + w_\mathrm{d}^2(\gamma^Y_t)|\dot\gamma^X_t|^2}\,\mathrm{d}t$$

holds.

Then we can define the metric d_w using this length structure:

Definition 3.10 (Warped product of metric spaces)**.** With the same assumptions of Definition 3.9, we define a pseudo-metric d_w on the space $Y \times X$ by

$$\mathrm{d}_w(p, q) := \inf\{l_w[\gamma] : \gamma \text{ is an absolutely continuous curve from } p \text{ to } q\},$$

for any $p, q \in Y \times X$.

d_w induces an equivalent relation on $Y \times X$: two points p, q are declared equivalent provided $\mathrm{d}_w(p,q) = 0$. The completion of the quotient of $Y \times X$ via this equivalence relation will be denoted by $Y \times_w X$. Then d_w induces a distance on $Y \times_w X$ which we shall continue to denote as d_w. Abusing a bit the notation, we shall also denote the typical element of $Y \times_w X$ as (y, x) with $y \in Y$ and $x \in X$ (there is no abuse in doing this if $w_\mathrm{d}(y) > 0$ and points in the completion not coming from points in $Y \times X$ will be negligible w.r.t. the warped product of measures and the same holds for the set of (y, x) such that $w_\mathrm{d}(y) = 0$, see below).

Notice that by definition $(Y \times_w X, \mathrm{d}_w)$ is a complete, separable and length space.

When considering the warped product of two metric measure spaces, we shall need to fix two warping functions: one for the distance and another for the measure.

Definition 3.11 (Warped products of metric measure spaces)**.** Let $(X, \mathrm{d}_X, \mathfrak{m}_X)$, $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ be two metric measure spaces and $w_\mathrm{d}, w_\mathfrak{m} : Y \to \mathbb{R}^+$ two functions. We say that $w_\mathrm{d}, w_\mathfrak{m}$ are warping functions provided they are continuous and such that $\{w_\mathrm{d} = 0\} \subset \{w_\mathfrak{m} = 0\}$.

In this case, the measure $\mathfrak{m}_w$ is defined via the formula:

$$\int f(x)g(y)\,\mathrm{d}\mathfrak{m}_w(y,x) = \int \left(\int f(x)w_\mathfrak{m}(y)\,\mathrm{d}\mathfrak{m}_X(x)\right) g(y)\,\mathrm{d}\mathfrak{m}_Y(y), \tag{3.4}$$

for any Borel non-negative functions f and g.

The warped product of $(X, \mathrm{d}_X, \mathfrak{m}_X)$, $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ via the functions $w_\mathrm{d}, w_\mathfrak{m}$ is then defined as $(Y \times_w X, \mathrm{d}_w, \mathfrak{m}_w)$.

It is immediate to verify that the assumption $\{w_\mathrm{d} = 0\} \subset \{w_\mathfrak{m} = 0\}$ grants that formula (3.4) truly defines a Borel measure on the space $(Y \times_w X, \mathrm{d}_w)$.

3.3 The result

3.3.1 Cartesian product

Throughtout this section $(X, \mathrm{d}, \mathfrak{m})$ is a fixed complete, separable and length space and $I \subset \mathbb{R}$ a closed, possibly unbounded, interval. We are interested in studying the Cartesian product $(X_c, \mathrm{d}_c, \mathfrak{m}_c)$ of I, endowed with its Euclidean structure, and $(X, \mathrm{d}, \mathfrak{m})$.

Given a function $f : X_c \to \mathbb{R}$ and $x \in X$ we denote by $f^{(x)} : I \to \mathbb{R}$ the function given by $f^{(x)}(t) := f(t, x)$. Similarly, for $t \in I$ we denote by $f^{(t)} : X \to \mathbb{R}$ the function given by $f^{(t)}(x) := f(t, x)$.

We start introducing the Beppo Levi space $\mathsf{BL}(X_c)$:

Definition 3.12 (The space $\mathsf{BL}(X_c)$)**.** The space $\mathsf{BL}(X_c) \subset L^2(X_c, \mathfrak{m}_c)$ is the space of functions $f \in L^2(X_c, \mathfrak{m}_c)$ such that

i) $f^{(x)} \in W^{1,2}(I)$ for $\mathfrak{m}$-a.e. x,

ii) $f^{(t)} \in W^{1,2}(X)$ for $\mathcal{L}^1$-a.e. t

iii) the function

$$|\mathrm{D}f|_c(t,x) := \sqrt{|\mathrm{D}f^{(t)}|_X^2(x) + |\mathrm{D}f^{(x)}|_I^2(t)},$$

belongs to $L^2(X_c, \mathfrak{m}_c)$.

On $\mathsf{BL}(X_c)$ we put the norm

$$\|f\|^2_{\mathsf{BL}(X_c)} := \|f\|^2_{L^2(X_c)} + \||\mathrm{D}f|_c\|^2_{L^2(X_c)}.$$

The space $\mathsf{BL}_{loc}(X_c)$ is the subset of $L^2_{loc}(X_c, \mathfrak{m}_c)$ of functions which are locally equal to some function in $\mathsf{BL}(X_c)$.

The main result of this section is the identification of the spaces $W^{1,2}(X_c)$ and $\mathsf{BL}(X_c)$ and of their corresponding weak gradients $|\mathrm{D}f|_{X_c}$ and $|\mathrm{D}f|_c$.

One inclusion has been proved in [9]:

Proposition 3.13 (Proposition 6.18 of [9])**.** *We have* $W^{1,2}(X_c) \subset \mathsf{BL}(X_c)$ *and*

$$\int_{X_c} |\mathrm{D}f|_c^2 \, \mathrm{d}\mathfrak{m}_c \leq \int_{X_c} |\mathrm{D}f|^2_{X_c} \, \mathrm{d}\mathfrak{m}_c, \qquad \forall f \in W^{1,2}(X_c). \tag{3.5}$$

To prove the other one it is useful to introduce the following classes of functions:

Definition 3.14 (The classes $\mathcal{A}$ and $\tilde{\mathcal{A}}$)**.** We define the space of functions $\mathcal{A} \subset \mathsf{BL}_{loc}(X_c)$ as

$$\mathcal{A} := \Big\{ g_1(x) + h(t)g_2(x) \in \mathsf{BL}_{loc}(X_c) \ : \ g_1, g_2 \in W^{1,2}(X), \quad h : I \to \mathbb{R} \ \text{ is Lipschitz} \Big\},$$

and the space $\tilde{\mathcal{A}} \subset \mathsf{BL}_{loc}(X_c)$ as the set of functions $f \in \mathsf{BL}_{loc}(X_c)$ which are locally equal to some function in $\mathcal{A}$.

Notice that Proposition 3.8 and the calclulus rules (3.2), (3.3) ensure that

$$\tilde{\mathcal{A}} \subset S^2_{loc}(X_c). \tag{3.6}$$

We start with the following purely metric lemma:

Lemma 3.15. *Let $f : X_c \to \mathbb{R}$ be of the form $f(t,x) = g_1(x) + h(t)g_2(x)$ for Lipschitz functions g_1, g_2, h. Then*

$$\mathrm{lip}(f)^2(t,x) \leq \mathrm{lip}_X(f^{(t)})^2(x) + \mathrm{lip}_I(f^{(x)})^2(t)$$

for every $(t,x) \in X_c$.

Proof. Let $(t,x), (s,y) \in X_c$, and notice that

$$\begin{aligned}
&|f(s,y) - f(t,x)| \\
&= |g_1(y) + h(s)g_2(y) - g_1(x) - h(t)g_2(x)| \\
&\leq |h(s) - h(t)||g_2(y)| + |g_1(y) - g_1(x) + h(t)(g_2(y) - g_1(x))| \\
&\leq \frac{|h(s) - h(t)||g_2(y)|}{|s-t|}|s-t| + \frac{|g_1(y) - g_1(x) + h(t)(g_2(y) - g_1(x))|}{\mathrm{d}(x,y)}\mathrm{d}(x,y) \\
&\leq \sqrt{\frac{|h(s) - h(t)|^2|g_2(y)|^2}{|s-t|^2} + \frac{|g_1(y) - g_1(x) + h(t)(g_2(y) - g_1(x))|^2}{\mathrm{d}^2(x,y)}}\,\mathrm{d}_c\big((s,y),(t,x)\big).
\end{aligned}$$

Dividing by $\mathrm{d}_c\big((s,y),(t,x)\big)$, letting $(s,y) \to (t,x)$ and using the continuity of g_2 we get the conclusion. □

The interest of functions in $\tilde{\mathcal{A}}$ is due to the next two results:

Proposition 3.16. *Let $f \in \tilde{\mathcal{A}}$. Then*

$$|\mathrm{D}f|_{X_c} = |\mathrm{D}f|_c \qquad \mathfrak{m}_c - \ a.e..$$

Proof. Notice that by (3.6) the statement makes sense. Moreover, due to the local nature of the statement we can assume that $f(t,x) = g_1(x) + h(t)g_2(x) \in \mathcal{A}$ with h having compact support. With this assumption we have that $f \in W^{1,2}(X_c)$ so that keeping in mind Proposition 3.13, to conclude it is sufficient to prove that

$$|\mathrm{D}f|^2_{X_c}(t,x) \leq |\mathrm{D}f^{(x)}|^2_I(t) + |\mathrm{D}f^{(t)}|^2_X(x), \qquad \mathfrak{m}_c - \ \text{a.e. } (t,x). \tag{3.7}$$

To this aim, it is in turn sufficient to show that for any $[a,b) \subset I$ and any Borel set $E \subset X$ we have

$$\int_{\tilde{E}} |\mathrm{D}f|^2_{X_c}(t,x)\,\mathrm{d}t\,\mathrm{d}\mathfrak{m}(x) \leq \int_{\tilde{E}} |\mathrm{D}f^{(x)}|^2_I(t) + |\mathrm{D}f^{(t)}|^2_X(x)\,\mathrm{d}t\,\mathrm{d}\mathfrak{m}(x) \tag{3.8}$$

with $\tilde{E} := [a,b) \times E$. Indeed if this holds, taking into account that open sets in X_c can always written as disjoint countable union of sets of the form $[a,b) \times E$, we deduce that

(3.8) holds with $\tilde{E}$ generic open set in X_c, so that using the fact that the integrand are in $L^1(X_c, \mathfrak{m}_c)$ by exterior approximation we get that (3.8) holds for arbitrary Borel sets $\tilde{E} \subset X_c$ and thus (3.7) and the conclusion.

Thus fix $E \subset X$ Borel, let $\tilde{E} := [a, b) \times E$ and up to a simple scaling argument assume also that $[a, b) = [0, 1)$.

For $k, i \in \mathbb{N}$, $k > 0$, we define $f_{k,i} \in W^{1,2}(X)$ as $f_{k,i}(x) := g_1(x) + h(\frac{i}{k})g_2(x)$ and $f_k \in \mathsf{BL}(X_c)$ as

$$f_k(t, x) := (kt - i)f_{k,i+1}(x) + (i + 1 - kt)f_{k,i}(x), \qquad \text{for } t \in \left[\tfrac{i}{k}, \tfrac{i+1}{k}\right].$$

Notice that $f_k \to f$ in $L^2(X_c, \mathfrak{m}_x)$. By Proposition 3.6, for each (k, i) we can find a sequence of Lipschitz functions $f_{k,i,n} \in \mathrm{Lip}\,(X)$ converging to $f_{k,i}$ in $L^2(X, \mathfrak{m})$ such that $\lim_{n\to\infty} \mathrm{lip}(f_{k,i,n}) = |\mathrm{D}f_{k,i}|_X$ in $L^2(X, \mathfrak{m})$.

Then we define $F_{k,n} \in \mathrm{Lip}(X_c)$ as

$$F_{k,n}(t, x) := (kt - i)f_{k,i+1,n}(x) + (i + 1 - kt)f_{k,i,n}(x), \qquad \text{for } t \in \left[\tfrac{i}{k}, \tfrac{i+1}{k}\right].$$

By construction we have $F_{k,n} \in \tilde{\mathcal{A}}$, so that Lemma 3.15 gives

$$|\mathrm{lip}(F_{k,n})|^2 \le |\mathrm{lip}_X(F_{k,n})|^2 + |\mathrm{lip}_I(F_{k,n})|^2, \qquad \mathcal{L}^1 \times \mathfrak{m} - \text{ a.e.},$$

moreover, since $\lim_{n\to\infty} F_{k,n} = f_k$ in $L^2(X_c, \mathfrak{m}_c)$ for every k, the lower semicontinuity of minimal weak upper gradients gives that

$$\int_{\tilde{E}} |\mathrm{D}f|^2_{X_c} \,\mathrm{d}\mathfrak{m}_c \le \varlimsup_{k\to\infty} \varlimsup_{n\to\infty} \int_{\tilde{E}} \mathrm{lip}_X(F_{k,n})^2 \,\mathrm{d}\mathfrak{m}_c + \varlimsup_{k\to\infty} \varlimsup_{n\to\infty} \int_{\tilde{E}} \mathrm{lip}_I(F_{k,n})^2 \,\mathrm{d}\mathfrak{m}_c. \tag{3.9}$$

Another direct consequence of the definition is that

$$\lim_{k\to\infty} \lim_{n\to\infty} \mathrm{lip}_I(F^{(x)}_{k,n})(t) = |\mathrm{D}f^{(x)}|_I(t), \qquad \mathfrak{m}_c - \text{ a.e. } (t, x),$$

which together with an application of the dominate convergence theorem grants that

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_{\tilde{E}} \mathrm{lip}_I(F^{(x)}_{k,n})^2(t) \,\mathrm{d}\mathfrak{m}_c(t, x) = \int_{\tilde{E}} |\mathrm{D}f^{(x)}|^2_I(t) \,\mathrm{d}\mathfrak{m}_c(t, x). \tag{3.10}$$

On the other hand, the continuity of h grants that $\mathbb{R} \ni t \mapsto f^{(t)} \in W^{1,2}(X)$ is continuous so that also the map $I \ni t \mapsto \int_E |\mathrm{D}f^{(t)}|^2_X \,\mathrm{d}\mathfrak{m}$ is continuous. In particular, its integral on

$[0,1]$ and coincides with the limit of the Riemann sums:

$$\begin{aligned}\int_{\tilde{E}} |\mathrm{D} f^{(t)}|_X^2(x)\, \mathrm{d}\mathfrak{m}_c(t,x) &= \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k} \int_E |\mathrm{D} f_{k,i}|_X^2 \,\mathrm{d}\mathfrak{m} \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{k} \sum_{i=0}^{k} \int_E \mathrm{lip}_X(f_{k,i,n})^2 \,\mathrm{d}\mathfrak{m}.\end{aligned} \tag{3.11}$$

From the very definition of $F_{k,n}$ we get that

$$\begin{aligned}\mathrm{lip}_X(F_{k,n}^{(t)})^2 &\le \big((kt-i)\mathrm{lip}_X(f_{k,i+1,n}) + (i+1-kt)\mathrm{lip}_X(f_{k,i,n})\big)^2 \\ &\le (kt-i)\mathrm{lip}_X(f_{k,i+1,n})^2 + (i+1-kt)\mathrm{lip}_X(f_{k,i,n})^2,\end{aligned}$$

on X for every $t \in [\frac{i}{k}, \frac{i+1}{k}]$, and thus

$$\begin{aligned}\int_{\tilde{E}} \mathrm{lip}_X(F_{k,n}^{(t)})^2(x)\,\mathrm{d}\mathfrak{m}_c(t,x) &\le \int_X \frac{1}{k}\sum_{i=0}^{k} \mathrm{lip}_X(f_{k,i,n})^2 - \frac{1}{2}\Big(\mathrm{lip}_X(f_{k,0,n})^2 + \mathrm{lip}_X(f_{k,k,n})^2\Big)\,\mathrm{d}\mathfrak{m} \\ &\le \int_X \frac{1}{k}\sum_{i=0}^{k} \mathrm{lip}_X(f_{k,i,n})^2\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

This inequality together with (3.11) give

$$\varlimsup_{k\to\infty}\varlimsup_{n\to\infty} \int_{\tilde{E}} \mathrm{lip}_X(F_{k,n}^{(t)})^2(x)\,\mathrm{d}\mathfrak{m}_c(t,x) \le \int_{\tilde{E}} |\mathrm{D} f^{(t)}|_X^2 \,\mathrm{d}t\,\mathrm{d}\mathfrak{m},$$

which together with (3.10) and (3.9) gives (3.8) and the conclusion. □

Proposition 3.17 (Density in energy)**.** *For any function $f \in \mathsf{BL}(X_c)$ there exists a sequence $(f_n) \subset \mathsf{BL}(X_c) \cap \mathcal{A}_{loc}$ converging to f in $L^2(X_c,\mathfrak{m}_c)$ such that $|\mathrm{D} f_n|_c \to |\mathrm{D} f|_c$ in $L^2(X_c,\mathfrak{m}_c)$ as $n\to\infty$.*

Proof. We shall give the proof for the case $I = \mathbb{R}$, the argument for arbitrary I being similar.

With a standard cut-off, truncation and diagonalization argument we can, and will, assume that the given $f \in \mathsf{BL}(X_c)$ is bounded and with bounded support. Then for any $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ we define

$$g_{i,n}(x) := n \int_{\frac{i}{n}}^{\frac{(i+1)}{n}} f(x,s)\,ds,$$

and

$$h_{i,n}(t) := \chi_n\big(t - \frac{i}{n}\big),$$

where $\chi_n : \mathbb{R} \mapsto \mathbb{R}$ is given by:

$$\chi_n(t) := \begin{cases} 0, & \text{if} \quad t < -\frac{1}{n}, \\ nt+1, & \text{if} \quad -\frac{1}{n} \leq t < 0, \\ 1-nt, & \text{if} \quad 0 \leq t < \frac{1}{n}, \\ 0, & \text{if} \quad \frac{1}{n} < t. \end{cases} \tag{3.12}$$

Then we define the sequence (f_n) as:

$$f_n(t,x) := \sum_{i \in \mathbb{Z}} h_{i,n}(t) g_{i,n}(x),$$

the sum being well defined because $g_{i,n}$ is not zero only for a finite number of i's and it is immediate to check that $f_n \in \tilde{\mathcal{A}}$.

We claim that $f_n \to f$ in $L^2(X_c, \mathfrak{m}_c)$ as $n \to \infty$. Integrating the inequality

$$\begin{aligned} \big(f_n(t,x)\big)^2 &= \left(\sum_{i \in \mathbb{Z}} h_{i,n}(t) g_{i,n}(x)\right)^2 \\ &\leq \sum_{i \in \mathbb{Z}} h_{i,n}(t) \big(g_{i,n}(x)\big)^2 \leq \sum_{i \in \mathbb{Z}} h_{i,n}(t) n \int_{i/n}^{(i+1)/n} f^2(s,x)\, \mathrm{d}s, \end{aligned}$$

on x and t we obtain $\|f_n\|_{L^2(X_c)} \leq \|f\|_{L^2(X_c)}$, for every $n \in \mathbb{N}$. This means that the linear operator T_n from $L^2(X_c, \mathfrak{m}_c)$ into itself assigning f_n to f is 1-Lipschitz for every $n \in \mathbb{N}$. Since obviously $f_n \to f$ in $L^2(X_c, \mathfrak{m}_c)$ if f is Lipschitz with bounded support, the uniform continuity of the T_n's grant that $f_n \to f$ in $L^2(X_c, \mathfrak{m}_c)$ for every $f \in L^2(X_c, \mathfrak{m}_c)$.

Now, taking into account the L^2-lower semicontinuity of the BL-norm, to conclude it is sufficient to show that for every $n \in \mathbb{N}$ we have

$$\begin{aligned} \int_{X_c} |\mathrm{D} f_n^{(t)}|_X^2(x)\, \mathrm{d}\mathfrak{m}_c(t,x) &\leq \int_{\mathbb{R}\times X} |\mathrm{D} f^{(t)}|_X^2(x)\, \mathrm{d}\mathfrak{m}_c(t,x), \\ \int_{X_c} |\mathrm{D} f_n^{(x)}|_{\mathbb{R}}^2(t)\, \mathrm{d}\mathfrak{m}_c(t,x) &\leq \int_{\mathbb{R}\times X} |\mathrm{D} f^{(x)}|_{\mathbb{R}}^2(x)\, \mathrm{d}\mathfrak{m}_c(t,x). \end{aligned} \tag{3.13}$$

Start noticing that the definition of the functions $g_{i,n}$, the convexity of minimal weak upper gradients and their L^2-lower semicontinuity yields that $g_{i,n} \in W^{1,2}(X)$ for every i, n with

$$\int_X |\mathrm{D} g_{i,n}|_X^2\, \mathrm{d}\mathfrak{m} \leq n \int_X \int_{i/n}^{(i+1)/n} |\mathrm{D} f^{(t)}|_{X'}^2\, \mathrm{d}t\, \mathrm{d}\mathfrak{m}. \tag{3.14}$$

Then from the trivial identity

$$f_n^{(t)} = (1+i-nt) g_{i,n} + (nt-i) g_{i+1,n},$$

valid for every n and a.e. $t \in [\frac{i}{n}, \frac{i+1}{n}]$ we know that $f_n^{(t)} \in W^{1,2}(X)$ and

$$\begin{aligned}|\mathrm{D} f_n^{(t)}|_X^2 &\le \big((1+i-nt)|\mathrm{D} g_{i,n}|_X + (nt-i)|\mathrm{D} g_{i+1,n}|_X\big)^2 \\ &\le (1+i-nt)|\mathrm{D} g_{i,n}|_X^2 + (nt-i)|\mathrm{D} g_{i+1,n}|_X^2,\end{aligned}$$

for every n and a.e. $t \in [\frac{i}{n}, \frac{i+1}{n}]$. This yields the bound

$$\begin{aligned}\int_{X_c} |\mathrm{D} f_n^{(t)}|_X^2(x)\,\mathrm{d}\mathfrak{m}_c(t,x) &\le \frac{1}{n}\sum_{i\in\mathbb{Z}}\int_X |\mathrm{D} g_{i,n}|_X^2(x)\,\mathrm{d}\mathfrak{m}(x) \\ \text{by (3.14)} \qquad &\le \sum_{i\in\mathbb{Z}}\int_X\int_{i/n}^{(i+1)/n} |\mathrm{D} f^{(t)}|_X^2(x)\,\mathrm{d}t\,\mathrm{d}\mathfrak{m}(x) \\ &= \int_{X_c} |\mathrm{D} f^{(t)}|_X^2(x)\,\mathrm{d}\mathfrak{m}_c(t,x),\end{aligned} \tag{3.15}$$

which is the first in (3.13).

Similarly, for $\mathfrak{m}$-a.e. $x \in X$ the function $f_n^{(x)} : \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}^1$-a.e. well defined and given by

$$f_n^{(x)}(t) = (1+i-nt)g_{i,n}(x) + (nt-i)g_{i+1,n}(x), \qquad \mathcal{L}^1-\text{ a.e. } t \in \big[\tfrac{i}{n}, \tfrac{i+1}{n}\big].$$

Arguing as before we get that $f_n^{(x)} \in W^{1,2}(\mathbb{R})$ for $\mathfrak{m}$-a.e. x and

$$\begin{aligned}\int_{i/n}^{(i+1)/n} |\mathrm{D} f_n^{(x)}|_{\mathbb{R}}^2(t)\,\mathrm{d}t &= \int_{i/n}^{(i+1)/n} n^2\big(g_{i+1,n}(x) - g_{i,n}(x)\big)^2\,\mathrm{d}t \\ &= n\big(g_{i+1,n}(x) - g_{i,n}(x)\big)^2 \\ &= n^3\left(\int_{(i+1)/n}^{(i+2)/n} f(t,x)\,\mathrm{d}t - \int_{i/n}^{(i+1)/n} f(t,x)\,\mathrm{d}t\right)^2 \\ &= n^3\left(\int_{i/n}^{(i+1)/n} f^{(x)}(t+1/n) - f^{(x)}(t)\,\mathrm{d}t\right)^2 \\ &\le n^3\left(\int_{i/n}^{(i+1)/n}\int_t^{t+1/n} |\mathrm{D} f^{(x)}|_{\mathbb{R}}(s)\,\mathrm{d}s\,\mathrm{d}t\right)^2 \\ &\le n\int_{i/n}^{(i+1)/n}\int_t^{t+1/n} |\mathrm{D} f^{(x)}|_{\mathbb{R}}^2(s)\,\mathrm{d}s\,\mathrm{d}t,\end{aligned}$$

which yields

$$\int_{X_c} |\mathrm{D} f_n^{(x)}|_{\mathbb{R}}^2(t)\,\mathrm{d}\mathfrak{m}_c(t,x) \le \int_{X_c} |\mathrm{D} f^{(x)}|_{\mathbb{R}}^2(t)\,\mathrm{d}\mathfrak{m}(t,x),$$

which is the second in (3.13) and the conclusion. □

We now have all the tools to prove the main result of this section:

Theorem 3.18. *The sets* $W^{1,2}(X_c)$ *and* $\mathsf{BL}(X_c)$ *coincide and for every* $f \in W^{1,2}(X_c) = \mathsf{BL}(X_c)$ *the identity*

$$|\mathrm{D}f|_{X_c} = |\mathrm{D}f|_c \qquad \mathfrak{m}_c - \ a.e.,$$

holds.

Proof. Proposition 3.13 gives the inclusion $W^{1,2}(X_c) \subset \mathsf{BL}(X_c)$. Now pick $f \in \mathsf{BL}(X_c)$ and find a sequence $(f_n) \subset \mathsf{BL}(X_c) \cap \mathcal{A}_{loc}$ as in Proposition 3.17. By Proposition 3.16 we know that

$$|\mathrm{D}f_n|_{X_c} = |\mathrm{D}f_n|_c \qquad \mathfrak{m}_c - \ \text{a.e.}, \ \forall n \in \mathbb{N}.$$

By construction, the right hand side converges to $|\mathrm{D}f|_c$ in $L^2(X_c, \mathfrak{m}_c)$ as $n \to \infty$, and since $f_n \to f$ in $L^2(X_c, \mathfrak{m}_c)$, by the lower semicontinuity of weak upper gradients we deduce that $f \in W^{1,2}(X_c)$ and

$$|\mathrm{D}f|_{X_c} \leq |\mathrm{D}f|_c, \qquad \mathfrak{m}_c - \ \text{a.e.},$$

which together with inequality (3.5) gives the thesis. □

3.3.2 Warped product

Throughtout this section $w_{\mathrm{d}}, w_{\mathfrak{m}} : I \to \mathbb{R}^+$ are given warping functions as in Definition 3.11. We are interested in studying Sobolev functions on the warped product space $(X_w, \mathrm{d}_w, \mathfrak{m}_w)$, where $X_w := I \times_w X$.

Like in the Cartesian case, given $f : X_w \to \mathbb{R}$ and $t \in I$ we shall denote by $f^{(t)} : X \to \mathbb{R}$ the function given by $f^{(t)}(x) := f(t, x)$. Similarly $f^{(x)}(t) := f(t, x)$ for $x \in X$.

We then consider the Beppo-Levi space $\mathsf{BL}(X_w)$ defined as follows:

Definition 3.19 (The space $\mathsf{BL}(X_w)$)**.** As a set, $\mathsf{BL}(X_w)$ is the subset of $L^2(X_w, \mathfrak{m}_w)$ made of those functions f such that:

i) for $\mathfrak{m}$-a.e. $x \in X$ we have $f^{(x)} \in W^{1,2}(\mathbb{R}, w_{\mathfrak{m}}\mathcal{L}^1)$,

ii) for $w_{\mathfrak{m}}\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have $f^{(t)} \in W^{1,2}(X)$,

iii) the function

$$|\mathrm{D}f|_w(t, x) := \sqrt{w_{\mathrm{d}}^{-2}(t)|\mathrm{D}f^{(t)}|_X^2(x) + |\mathrm{D}f^{(x)}|_{\mathbb{R}}^2(t)} \tag{3.16}$$

belongs to $L^2(X_w, \mathfrak{m}_w)$.

On $\mathsf{BL}(X_w)$ we put the norm

$$\|f\|_{\mathsf{BL}(X_w)} := \sqrt{\|f\|^2_{L^2(X_w)} + \||\mathrm{D}f|_w\|^2_{L^2(X_w)}}.$$

It will be useful to introduce the following auxiliary space:

Definition 3.20 (The space $\mathsf{BL}_0(X_w)$)**.** Let $V \subset \mathsf{BL}(X_w)$ be the space of functions f which are identically 0 on $\Omega \times X \subset X_w$ for some open set $\Omega \subset \mathbb{R}$ containing $\{w_{\mathfrak{m}} = 0\}$.

$\mathsf{BL}_0(X_w) \subset \mathsf{BL}(X_w)$ is defined as the closure of V in $\mathsf{BL}(X_w)$.

The goal of this section is to compare the spaces $\mathsf{BL}(X_w)$ and $W^{1,2}(X_w)$ and their respective notions of minimal weak upper gradients, namely $|\mathrm{D}f|_w$ and $|\mathrm{D}f|_{X_w}$. Under the sole continuity assumption of $w_{\mathrm{d}}, w_{\mathfrak{m}}$ and the compatibility condition $\{w_{\mathrm{d}} = 0\} \subset \{w_{\mathfrak{m}} = 0\}$ we can prove that

$$\mathsf{BL}_0(X_w) \subset W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$$

and that for any $f \in W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$ the identity

$$|\mathrm{D}f|_{X_w} = |\mathrm{D}f|_w$$

holds $\mathfrak{m}_w$-a.e., so that in particular the above inclusions are continuous. Without additional hypotheses it is unclear to us whether $W^{1,2}(X_w) = \mathsf{BL}(X_w)$ (on the other hand, it is easy to construct examples where $\mathsf{BL}_0(X_w)$ is strictly smaller than $\mathsf{BL}(X_w)$). Still, if we assume that

$$\text{the set } \{w_{\mathfrak{m}} = 0\} \subset I \text{ is discrete} \tag{3.17}$$

and that $w_{\mathfrak{m}}$ decays at least linearly near its zeros, i.e.

$$w_{\mathfrak{m}}(t) \leq C \inf_{s: w_{\mathfrak{m}}(s)=0} |t-s|, \qquad \forall t \in \mathbb{R}, \tag{3.18}$$

for some constant $C \in \mathbb{R}$, then we can prove - using basically arguments about capacities - that

$$\mathsf{BL}_0(X_w) = \mathsf{BL}(X_w),$$

so that the three spaces considered are all equal. We remark that these two additional assumptions on $w_{\mathfrak{m}}$ are satisfied in all the geometric applications we have in mind, because typically one considers cone/spherical suspensions and in these cases $w_{\mathfrak{m}}$ has at most two zeros and decays polynomially near them.

We turn to the details. The following result is easily established:

Proposition 3.21. *Let $w_{\mathrm{d}}, w_{\mathfrak{m}}$ be warping functions. Then $W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$.*

Proof. Pick $f \in W^{1,2}(X_w)$ and use Proposition 3.6 to find a sequence (f_n) of Lipschitz functions on X_w such that $f_n \to f$ and $\mathrm{lip}(f_n) \to |\mathrm{D}f|_{X_w}$ in $L^2(X_w)$. Up to pass to a fast converging subsequence, not relabeled, we can further assume that for $\mathfrak{m}$-a.e. $x \in X$, we have $f_n^{(x)} \to f^{(x)}$ in $L^2(I, w_{\mathfrak{m}}\mathcal{L}^1)$ and that for $w_{\mathfrak{m}}\mathcal{L}^1$-a.e. $t \in I$ we have $f_n^{(t)} \to f^{(t)}$ in $L^2(X, \mathfrak{m})$.

Observe that for every $(t, x) \in X_w$ we have

$$\begin{aligned}
\mathrm{lip}(f_n)(t,x) &= \varlimsup_{(s,y)\to(t,x)} \frac{|f_n(s,y) - f_n(t,x)|}{\mathrm{d}_w((s,y),(t,x))} \\
&\geq \varlimsup_{s\to t} \frac{|f_n(s,x) - f_n(t,x)|}{\mathrm{d}_w((s,x),(t,x))} \\
&= \varlimsup_{s\to t} \frac{|f_n^{(x)}(s) - f_n^{(x)}(t)|}{|s-t|} = \mathrm{lip}_{\mathbb{R}}(f_n^{(x)})(t)
\end{aligned}$$

and therefore by Fatou's lemma we deduce

$$\begin{aligned}
\int_X \varliminf_{n\to\infty} \int_I \mathrm{lip}_I(f_n^{(x)})^2(t)\,\mathrm{d}(w_{\mathfrak{m}}\mathcal{L}^1)(t)\,\mathrm{d}\mathfrak{m}(x) &\leq \varliminf_{n\to\infty} \int_{X_w} \mathrm{lip}(f_n)^2(t,x)\,\mathrm{d}\mathfrak{m}_w(t,x) \\
&= \int_{X_w} |\mathrm{D}f|^2_{X_w}\,\mathrm{d}\mathfrak{m}_w < \infty.
\end{aligned}$$

Since $f_n^{(x)} \to f^{(x)}$ in $L^2(I, w_{\mathfrak{m}}\mathcal{L}^1)$ for $\mathfrak{m}$-a.e. $x \in X$, this last inequality together with the lower semicontinuity of minimal weak upper gradients ensures that $f^{(x)} \in W^{1,2}(I, w_{\mathfrak{m}}\mathcal{L}^1)$ for $\mathfrak{m}$-a.e. $x \in X$ and

$$\int_{X_w} |\mathrm{D}f^{(x)}|_I^2(t)\,\mathrm{d}\mathfrak{m}_w(t,x) \leq \int_{X_w} |\mathrm{D}f|^2_{X_w}\,\mathrm{d}\mathfrak{m}_c. \tag{3.19}$$

An analogous argument starting from the bound

$$\begin{aligned}
\mathrm{lip}(f_n)(t,x) &= \varlimsup_{(s,y)\to(t,x)} \frac{|f_n(s,y) - f_n(t,x)|}{\mathrm{d}_w((s,y),(t,x))} \\
&\geq \varlimsup_{y\to x} \frac{|f_n(t,y) - f_n(t,x)|}{\mathrm{d}_w((t,y),(t,x))} \\
&= \varlimsup_{y\to x} \frac{|f_n^{(t)}(y) - f_n^{(t)}(x)|}{w(t)\mathrm{d}(x,y)} = \frac{1}{w(t)}\mathrm{lip}_I(f_n^{(t)})(x)
\end{aligned}$$

valid for every $t \in I$ such that $w_{\mathrm{d}}(t) > 0$, grants that $f^{(t)} \in W^{1,2}(X)$ for $w_{\mathfrak{m}}\mathcal{L}^1$-a.e. $t \in I$ (recall that $\{w_{\mathrm{d}} = 0\} \subset \{w_{\mathfrak{m}} = 0\}$) and

$$\int_{X_w} |\mathrm{D}f^{(t)}|_X^2(x)\,\mathrm{d}\mathfrak{m}_w(t,x) \leq \int_{X_w} |\mathrm{D}f|^2_{X_w}\,\mathrm{d}\mathfrak{m}_w. \tag{3.20}$$

The bounds (3.19) and (3.20) ensure that $f \in \mathsf{BL}(X_w)$, so that the inclusion $W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$ is proved. $\square$

In order to prove that for $f \in W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$ the minimal weak upper gradient $|\mathrm{D}f|_{X_w}$ coincides with the 'warped' gradient $|\mathrm{D}f|_w$ defined in (3.16), we shall make use of the following simple comparison argument, which will then allow us to reduce the proof to the already known cartesian case.

Lemma 3.22. *Let X be a set, $\mathrm{d}_1, \mathrm{d}_2$ two distances on it and $\mathfrak{m}_1, \mathfrak{m}_2$ two measures. Assume that $(X, \mathrm{d}_1, \mathfrak{m}_1)$ and $(X, \mathrm{d}_2, \mathfrak{m}_2)$ are both metric measure spaces satisfying the Assumptions 3.1, that for some $C > 0$ we have $\mathfrak{m}_2 \leq C\mathfrak{m}_1$ and that for some $L > 0$ we have $\mathrm{d}_1 \leq L\mathrm{d}_2$.*

Then denoting by $S(X_1), S(X_2)$ the Sobolev classes relative to $(X, \mathrm{d}_1, \mathfrak{m}_1)$ and $(X, \mathrm{d}_2, \mathfrak{m}_2)$ respectively and by $|\mathrm{D}f|_1, |\mathrm{D}f|_2$ the associated minimal weak upper gradients, we have

$$S(X_1) \subset S(X_2)$$

and for every $f \in S(X_1)$ the inequality

$$|\mathrm{D}f|_2 \leq L|\mathrm{D}f|_1,$$

holds $\mathfrak{m}_2$-a.e..

Proof. The assumptions ensure that the topology induced by d_2 is finer than the one induced by d_1, hence every d_1-Borel function is also d_2-Borel. Then observe that the assumption $\mathrm{d}_1 \leq L\mathrm{d}_2$ ensures that d_2-absolutely continuous curves are also d_1-absolutely continuous, the d_1-metric speed being bounded by L-times the d_2-metric speed. Then considering also the assumption $\mathfrak{m}_2 \leq C\mathfrak{m}_1$ we see that $(X, \mathrm{d}_2, \mathfrak{m}_2)$ -test plans are also $(X, \mathrm{d}_1, \mathfrak{m}_1)$-test plans, which, by definition, gives the inclusion $S(X_1) \subset S(X_2)$. The inequality $|\mathrm{D}f|_2 \leq L|\mathrm{D}f|_1$ $\mathfrak{m}_2$-a.e. is then obtained by the $\mathfrak{m}_2$-a.e. minimality of $|\mathrm{D}f|_2$ and the opposite inequality valid for the metric speeds. $\square$

We can then prove the following result:

Proposition 3.23. *Let $w_\mathrm{d}, w_\mathfrak{m}$ be warping functions and $f \in W^{1,2}(X_w) \subset \mathsf{BL}(X_w)$. Then*

$$|\mathrm{D}f|_{X_w} = |\mathrm{D}f|_w, \qquad \mathfrak{m}_w - \quad a.e..$$

Proof. Fix $\epsilon > 0$ and $t_0 \in \mathbb{R}$ such that $w_{\mathfrak{m}}(t_0) > 0$ so that also $w_{\mathrm{d}}(t_0) > 0$. Use the continuity of w_{d} to find $\delta > 0$ so that

$$\left|\frac{w_{\mathrm{d}}(t)}{w_{\mathrm{d}}(s)}\right| \leq 1+\epsilon \qquad \forall t, s \in [t_0 - 2\delta, t_0 + 2\delta] \tag{3.21}$$

and let $\chi : \mathbb{R} \to [0,1]$ be a Lipschitz function identically 1 on $[t_0 - \delta, t_0 + \delta]$ with support contained in $[t_0 - 2\delta, t_0 + 2\delta]$.

We introduce the continuous functions $\bar{w}_{\mathrm{d}}, \bar{w}_{\mathfrak{m}} : \mathbb{R} \to \mathbb{R}$ as

$$\bar{w}_{\mathrm{d}}(t) := \begin{cases} w_{\mathrm{d}}(t_0 - 2\delta), & \text{if } t < t_0 - 2\delta, \\ w_{\mathrm{d}}(t), & \text{if } t \in [t_0 - 2\delta, t_0 + 2\delta], \\ w_{\mathrm{d}}(t_0 + 2\delta), & \text{if } t > t_0 + 2\delta, \end{cases}$$

$$\bar{w}_{\mathfrak{m}}(t) := \begin{cases} w_{\mathfrak{m}}(t_0 - 2\delta), & \text{if } t < t_0 - 2\delta, \\ w_{\mathfrak{m}}(t), & \text{if } t \in [t_0 - 2\delta, t_0 + 2\delta], \\ w_{\mathfrak{m}}(t_0 + 2\delta), & \text{if } t > t_0 + 2\delta, \end{cases}$$

the corresponding product space $(X_{\bar{w}}, \mathrm{d}_{\bar{w}}, \mathfrak{m}_{\bar{w}})$ and consider the function $\bar{f} : X_w \to \mathbb{R}$ given by $\bar{f}(t,x) := \chi(t) f(t,x)$ which belongs to $W^{1,2}(X_w)$ and therefore, by what we just proved, to $\mathsf{BL}(X_w)$. The locality property of minimal weak upper gradients ensure that

$$|\mathrm{D}f|_{X_w} = |\mathrm{D}\bar{f}|_{X_w} \quad \text{and} \quad |\mathrm{D}f|_w = |\mathrm{D}\bar{f}|_w \qquad \mathfrak{m}_w - \text{ a.e. on } [t_0 - \delta, t_0 + \delta] \times X.$$

Since $\bar{f}$ has support concentrated in the set of (t,x)'s with $t \in [t_0 - 2\delta, t_0 + 2\delta]$ and w_{d} is positive in such interval, we can think at $\bar{f}$ also as a real valued function $X_{\bar{w}}$. With this identification in mind it is clear that

$$|\mathrm{D}\bar{f}|_{X_w} = |\mathrm{D}\bar{f}|_{X_{\bar{w}}} \quad \text{and} \quad |\mathrm{D}\bar{f}|_w = |\mathrm{D}\bar{f}|_{\bar{w}} \qquad \mathfrak{m}_w - \text{ a.e. on } [t_0 - 2\delta, t_0 + 2\delta] \times X.$$

We now consider the cartesian product $(X_c, \mathrm{d}_c, \mathfrak{m}_c)$ of $(X, \mathrm{d}, \mathfrak{m})$ and $\mathbb{R}$. Notice that the sets $X_{\bar{w}}$ and X_c both coincide with $\mathbb{R} \times X$ and that by construction (recall also (3.21)) we have

$$c\mathfrak{m}_c \leq \mathfrak{m}_{\bar{w}} \leq C\mathfrak{m}_c \qquad \text{and} \qquad \frac{w_{\mathrm{d}}(t_0)}{1+\varepsilon}\mathrm{d}_c \leq \mathrm{d}_{\bar{w}} \leq w_{\mathrm{d}}(t_0)(1+\varepsilon)\mathrm{d}_c$$

for some $c, C > 0$. Hence by Lemma 3.22 we deduce that $\mathfrak{m}_{\bar{w}}$-a.e. it holds

$$\frac{|\mathrm{D}\bar{f}|_{X_c}}{w_{\mathrm{d}}(t_0)(1+\varepsilon)} \leq |\mathrm{D}\bar{f}|_{X_{\bar{w}}} \leq \frac{1+\varepsilon}{w_{\mathrm{d}}(t_0)}|\mathrm{D}\bar{f}|_{X_c} \quad \text{and} \quad \frac{|\mathrm{D}\bar{f}|_c}{w_{\mathrm{d}}(t_0)(1+\varepsilon)} \leq |\mathrm{D}\bar{f}|_{\bar{w}} \leq \frac{1+\varepsilon}{w_{\mathrm{d}}(t_0)}|\mathrm{D}\bar{f}|_c.$$

Since by Theorem 3.18 we know that $|\mathrm{D}\bar{f}|_{X_c} = |\mathrm{D}\bar{f}|_c$ $\mathfrak{m}_c$-a.e., collecting what we proved we deduce that

$$\frac{|\mathrm{D}f|_{X_w}}{(1+\varepsilon)^2} \leq |\mathrm{D}f|_w \leq (1+\varepsilon)^2 |\mathrm{D}f|_{X_w}$$

$\mathfrak{m}_w$-a.e. on $[t_0-\delta, t_0+\delta]\times X$. By the arbitrariness of t_0 such that $w_{\mathfrak{m}}(t_0)>0$ and the Lindelof property of $\{w_{\mathfrak{m}}>0\}\subset\mathbb{R}$ we deduce that the above inequality holds $\mathfrak{m}_w$-a.e.. The conclusion then follows letting $\varepsilon\downarrow 0$. $\square$

We now turn to the general relation between $\mathsf{BL}_0(X_w)$ and $W^{1,2}(X_w)$:

Proposition 3.24. *Let $w_{\mathrm{d}}, w_{\mathfrak{m}}$ be warping functions. Then $\mathsf{BL}_0(X_w)\subset W^{1,2}(X_w)$.*

Proof. Taking into account Proposition 3.23 it is sufficient to prove that $V\subset W^{1,2}(X_w)$. Notice that for arbitrary $f\in\mathsf{BL}(X_w)$, considering the functions $\chi_n(t) := 0\vee(n-|t|)\wedge 1$ and defining $f_n(t,x) := \chi_n(t)f(t,x)$, via a direct verification of the definitions we have $f_n\in\mathsf{BL}(X_w)$, while inequality (3.3) and the dominate convergence theorem grant that $f_n\to f$ in $\mathsf{BL}(X_w)$. Therefore, using again Proposition 3.23 whcih ensures that BL-convergence implies $W^{1,2}$-convergence, to conclude it is sufficient to show that any $f\in V$ with support contained in $(I\cap[-T,T])\times X\subset X_w$ for some $T>0$ belongs to $W^{1,2}(X_w)$.

Thus fix such $f\in V$, for $r>0$ denote by $\Omega_r\subset\mathbb{R}$ the r-neighborhood of $\{w_{\mathfrak{m}}=0\}$ and find $r\in(0,1)$ such that f is $\mathfrak{m}_w$-a.e. zero on $\Omega_{2r}\times X$. Then by continuity and compactness and recalling that $\{w_{\mathrm{d}}=0\}\subset\{w_{\mathfrak{m}}=0\}$ we deduce that there are constants $0<c\leq C<\infty$ such that

$$c\leq w_{\mathrm{d}}(t), w_{\mathfrak{m}}(t)\leq C, \qquad \forall t\in I\cap[-T,T]\setminus\Omega_{r/2}.$$

We are now going to use a comparison argument similar to that used in the proof of Proposition 3.23. Find two continuous functions $w'_{\mathrm{d}}, w_{\mathfrak{m}}$ agreeing with $w_{\mathrm{d}}, w_{\mathfrak{m}}$ on $[-T,T]\setminus\Omega$ and such that $c\leq w'_{\mathrm{d}}, w'_{\mathfrak{m}}\leq C$ on the whole $\mathbb{R}$ and consider the warped product $(X_{w'}, \mathrm{d}_{w'}, \mathfrak{m}_{w'})$ and the cartesian product $(X_c, \mathrm{d}_c, \mathfrak{m}_c)$ of I and X. We then have the equalities of sets:

$$\mathsf{BL}(X_{w'}) = \mathsf{BL}(X_c) = W^{1,2}(X_c) = W^{1,2}(X_{w'}),$$

the first and last coming from Lemma 3.22 and the properties of $w'_{\mathrm{d}}, w'_{\mathfrak{m}}$ and the middle one being given by Theorem 3.18.

By the construction of $w'_{\mathrm{d}}, w'_{\mathfrak{m}}$ we see that $f\in\mathsf{BL}(X_{w'})$ and thus, by what we just proved, that $f\in W^{1,2}(X_{w'})$. Then Proposition 3.6 grants that there exists a sequence

(f_n) of $\mathrm{d}_{w'}$-Lipschitz functions converging to f in $L^2(X_{w'})$ with

$$\sup_{n\in\mathbb{N}} \int \mathrm{lip}'(f_n)^2 \,\mathrm{d}\mathfrak{m}_{w'} < \infty$$

uniformly bounded in n, where by lip' we denote the local Lipschitz constant computed w.r.t. the distance $\mathrm{d}_{w'}$. Notice that up to replacing f_n with $(-C_n) \vee f_n \wedge C_n$ for a sufficiently large C_n, we can, and will, assume that f_n is bounded for every $n \in \mathbb{N}$.

Now find a Lipschitz function $\chi : I \to [0,1]$ identically 0 on $\Omega_r \cup (I \setminus [-T-1, T+1])$, identically 1 on $I \cap [-T,T] \setminus \Omega_{2r}$ and put $\tilde{f}_n(t,x) := \chi(t) f_n(t,x)$. By construction it is immediate to check that the $\tilde{f}_n$'s are still $\mathrm{d}_{w'}$-Lipschitz, converging to f in $L^2(\mathfrak{m}_{w'})$ and satisfying

$$\sup_{n\in\mathbb{N}} \int \mathrm{lip}'(\tilde{f}_n)^2 \,\mathrm{d}\mathfrak{m}_{w'} < \infty. \tag{3.22}$$

We now claim that the $\tilde{f}_n$'s are d_w-Lipschitz, converging to f in $L^2(X_w)$ and such that

$$\sup_{n\in\mathbb{N}} \int \mathrm{lip}(\tilde{f}_n)^2 \,\mathrm{d}\mathfrak{m}_{w} < \infty, \tag{3.23}$$

from which the conclusion follows by the lower semicontinuity of weak upper gradients and the bound $\mathrm{lip}(f_n) \leq |\mathrm{D}f|_{X_w}$ valid $\mathfrak{m}_w$-a.e.. Since all the functions $\tilde{f}_n$ and f are concentrated on $([-T,T] \setminus \Omega_r) \times X$ and on this set the measures $\mathfrak{m}_w$ and $\mathfrak{m}_{w'}$ agree, we clearly have $L^2(X_w)$-convergence. Moreover, since w_d and $w_{\mathrm{d}'}$ agree on $([-T,T] \setminus \Omega_r) \times X$, the topologies on $([-T,T] \setminus \Omega_r) \times X$ induced by d_w and $\mathrm{d}_{w'}$ agree (with the product topology, given that these functions are positive) and a direct use of the definition yields

$$\lim_{(s,y)\to(t,x)} \frac{\mathrm{d}_w\big((s,y),(t,x)\big)}{\mathrm{d}_{w'}\big((s,y),(t,x)\big)} = 1, \qquad \forall (t,x) \in ([-T,T] \setminus \Omega_r) \times X.$$

In particular, we have $\mathrm{lip}(\tilde{f}_n) = \mathrm{lip}'(\tilde{f}_n)$ in $([-T,T] \setminus \Omega_r) \times X$, so that (3.23) follows from (3.22). Finally, recalling that a Borel function on $[0,1]$ whose local Lipschitz constant is uniformly bounded by some constant L is in fact L-Lipschitz (as shown by a direct covering argument) and using the fact that (X_w, d_w) is by definition a length space we see that for every $n \in \mathbb{N}$ it holds

$$\mathrm{Lip}(f_n) = \sup_{X_w} \mathrm{lip}(f_n) = \sup_{X_{w'}} \mathrm{lip}'(f_n) = \mathrm{Lip}'(f_n) < \infty,$$

where $\mathrm{Lip}'(f_n)$ denotes the $\mathrm{d}_{w'}$-Lipschitz constant. Hence f_n is d_w-Lipschitz for every $n \in \mathbb{N}$ and the proof is achieved. $\square$

Finally, we prove that if the set of zeros of $w_\mathfrak{m}$ is discrete and $w_\mathfrak{m}$ decays at least linearly close to its zeros, then $\mathsf{BL}_0(X_w) = \mathsf{BL}(X_w)$:

Proposition 3.25. *Let $w_{\mathrm{d}}, w_{\mathfrak{m}}$ be warping functions and assume that $w_{\mathfrak{m}}$ has the properties* (3.17) *and* (3.18).

Then $\mathsf{BL}_0(X_w) = \mathsf{BL}(X_w)$.

Proof. A standard truncation argument shows that $\mathsf{BL} \cap L^\infty(X_w)$ is dense in $\mathsf{BL}(X_w)$, so to conclude it is sufficient to show that for any $f \in \mathsf{BL} \cap L^\infty(X_w)$ we can find a sequence $(f_n) \subset V$ converging to it in $\mathsf{BL}(X_w)$.

Thus pick $f \in \mathsf{BL} \cap L^\infty(X_w)$, put $D(t) := \min_{s: w_{\mathfrak{m}}(s)=0} |t-s|$ and for $n, m \in \mathbb{N}$, $n > 1$ consider the cut-off functions

$$
\begin{aligned}
\sigma_m(x) &:= 0 \vee (m - \mathrm{d}(x, \bar{x})) \wedge 1, \\
\eta_n(t) &:= 0 \vee \Big(1 - \frac{|\log(D(t))|}{\log(n)}\Big) \wedge 1, \\
\tilde{\eta}_n(t) &:= 0 \vee \Big(n - |t|\Big) \wedge 1,
\end{aligned}
$$

where $\bar{x} \in X$ is a chosen, fixed point, and define $f_{n,m}(t,x) := \eta_n(t)\tilde{\eta}_n(t)\sigma_m(x)f(t,x)$. Since $(t,x) \mapsto \eta_n(t)\tilde{\eta}_n(t)\sigma_m(x)$ is Lipschitz and bounded for every n, m, a direct check of the definition of $\mathsf{BL}(X_w)$ shows that $f_{n,m} \in \mathsf{BL}(X_w)$ for every n, m and, since η_n is 0 on a neighborhood of $\{w_{\mathfrak{m}} = 0\}$, we also have $f_{n,m} \in V$ for every n, m.

Using the fact that the functions $(t,x) \mapsto \eta_n(t)\tilde{\eta}_n(t)\sigma_m(x)$ are uniformly bounded by 1 and pointwise converge to 1 as $n, m \to \infty$ and the dominate convergece theorem we see that $f_{n,m} \to f$ in $L^2(X_w)$ as $n, m \to \infty$.

Next, recalling (3.3) and using that σ_m is 1-Lipschitz we see that

$$
|\mathrm{D}(f^{(t)} - f^{(t)}_{n,m})|_X(x) \leq \big|\eta_n(t)\tilde{\eta}_n(t)\sigma_m(x) - 1\big|\,|\mathrm{D}f^{(t)}|_X(x) + |f(t,x)|1_{\{\mathrm{d}(\cdot,\bar{x}) \geq m-1\}}(x)
$$

for $\mathfrak{m}_w$-a.e. (t,x), so that the dominate convergence theorem again gives that $\int |\mathrm{D}(f^{(t)} - f^{(t)}_{n,m})|^2_X(x)\,\mathrm{d}\mathfrak{m}_w(t,x) \to 0$ as $n, m \to \infty$.

Similarly, we have

$$
\begin{aligned}
|\mathrm{D}(f^{(x)} - f^{(x)}_{n,m})|_I(t) \leq& \big|\eta_n(t)\tilde{\eta}_n(t)\sigma_m(x) - 1\big|\,|\mathrm{D}f^{(x)}|_I(t) + |f(t,x)|1_{\{|\cdot| \geq n-1\}}(t) \\
&+ |f(t,x)|1_{\{\mathrm{d}(\cdot,\bar{x}) \leq m\}}(x)1_{\{|\cdot| \leq n\}}(t)|\partial_t \eta_n|(t)
\end{aligned}
$$

for $\mathfrak{m}_w$-a.e. (t,x) and again by dominate convergence we see that the first two terms in the right hand side go to 0 in $L^2(X_w)$ as $n, m \to \infty$. For the last term, we use the fact that f is bounded and our assumptions on $w_{\mathfrak{m}}$. Observe indeed that $|\partial_t \eta_n|(t) \leq \frac{1_{D^{-1}([n^{-1},1])}(t)}{D(t)\log n}$

so that letting $x_1, \ldots, x_N$ be the finite number of zeros of $w_{\mathfrak{m}}$ in $[-n-1, n+1]$ we have

$$\begin{aligned}\int |f(t,x)|^2 1_{\{\mathrm{d}(\cdot,\bar{x})\leq m\}}(x) 1_{\{|\cdot|\leq n\}}(t) |\partial_t \eta_n|^2(t)\,\mathrm{d}\mathfrak{m}_w \\ &\leq \frac{\|f\|_{L^\infty}\mathfrak{m}(B_m(\bar{x}))}{\log(n)^2}\int_{[-n,n]\cap D^{-1}([n^{-1},1])}\frac{1}{D^2(t)}w_{\mathfrak{m}}(t)\,\mathrm{d}t \\ &\leq C\frac{\|f\|_{L^\infty}\mathfrak{m}(B_m(\bar{x}))}{\log(n)^2}\int_{[-n,n]\cap D^{-1}([n^{-1},1])}\frac{1}{D(t)}\,\mathrm{d}t \\ &\leq C\frac{\|f\|_{L^\infty}\mathfrak{m}(B_m(\bar{x}))}{\log(n)^2}\sum_{i=1}^{N}\int_{\{t:|t-x_i|\in[n^{-1},1]\}}\frac{1}{|t-x_i|}\,\mathrm{d}t \\ &= 2NC\frac{\|f\|_{L^\infty}\mathfrak{m}(B_m(\bar{x}))}{\log(n)}.\end{aligned}$$

Since the last term goes to 0 as $n \to \infty$ for every $m \in \mathbb{N}$, we just proved that

$$\lim_{m\to\infty}\lim_{n\to\infty}\int |\mathrm{D}(f^{(x)} - f^{(x)}_{n,m})|_I^2(t)\,\mathrm{d}\mathfrak{m}_w(t,x) = 0,$$

which is sufficient to conclude. □

3.3.3 Sobolev-to-Lipschitz property

We recall the following definition:

Definition 3.26 (Sobolev-to-Lipschitz property)**.** We say that a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ has Sobolev to Lipschitz property if for any function $f \in W^{1,2}(X)$ with $|\mathrm{D}f|_X \in L^\infty(X)$, we can find a function $\tilde{f}$ such that $f = \tilde{f}$ $\mathfrak{m}$-a.e. and $\mathrm{Lip}(f) = \operatorname{ess\,sup} |\mathrm{D}f|_X$.

Aim of this section is to study the Sobolev-to-Lipschitz property on warped products.

Metric measure spaces with the Sobolev-to-Lipschitz property are, in some sense, those whose metric properties can be studied via Sobolev calculus. Only quite regular metric measure structures possess this property (for instance, doubling& Poincaré are not sufficient to ensure the Sobolev-to-Lipschitz property) and it is a non-trivial fact that $\mathrm{RCD}(K,\infty)$ spaces have such property (see [9] for the definition of $\mathrm{RCD}(K,\infty)$ spaces and the proof of the claim).

The fact that $\mathrm{RCD}(K,\infty)$ spaces have such property is tightly linked to the following regularity result for geodesic interpolation in the space of probability measures:

Proposition 3.27. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(K,\infty)$ space and $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with bounded support and such that $\mu_0, \mu_1 \leq C\mathfrak{m}$ for some $C > 0$. Then there exists only one* *geodesic* (μ_t) *connecting* μ_0 *to* μ_1 *and such geodesic satisfies*

$$\mu_t \le C'\mathfrak{m}, \qquad \forall t \in [0,1], \tag{3.24}$$

for some $C' > 0$.

It was proved in [9] that the Cartesian product of two $\mathrm{RCD}(K,\infty)$ spaces is still $\mathrm{RCD}(K,\infty)$ and thus in particular it has the Sobolev-to-Lipschitz property. Here we consider the warped product of an $\mathrm{RCD}(K,\infty)$ space $(X,\mathrm{d},\mathfrak{m})$ and an interval I.

We observe that under the only assumption that $w_\mathrm{d}, w_\mathfrak{m}$ are warping functions we cannot hope to prove that X_w has the Sobolev-to-Lipschitz property. Indeed, if $w_\mathfrak{m}$ is 0 on some subinterval of I which disconnects I, then there are functions on X_w which are locally constant on the support of $\mathfrak{m}_w$, hence Sobolev with 0 weak upper gradient, without being constant on X_w.

We shall therefore only consider the case where $w_\mathfrak{m}$ is strictly positive in the interior of I, a condition which is satisfied in the standard geometric constructions like that of cone/spherical subspension.

We introduce the following auxiliary concept, which is a sort of length property which takes into account the reference measure:

Definition 3.28 (Good space)**.** We say that a metric measure space $(X,\mathrm{d},\mathfrak{m})$ is a good space if for some Borel subset $A \subset X$ of full $\mathfrak{m}$-measure and every $x,y \in A$ there exists a family of $\mathcal{W}_2$-absolutely continuous curves $t \mapsto \mu_{t,\epsilon} \in \mathcal{P}_2(X)$ indexed by a parameter $\epsilon > 0$ such that:

a) $\mu_{0,\epsilon} = \frac{1_{B_{r_\epsilon}(x)}}{\mathfrak{m}(B_{r_\epsilon}(x))}\mathfrak{m}$ and $\mu_{1,\epsilon} = \frac{1_{B_{r_\epsilon}(y)}}{\mathfrak{m}(B_{r_\epsilon}(y))}\mathfrak{m}$ for some $r_\epsilon > 0$ such that $\lim_{\epsilon\downarrow 0} r_\epsilon = 0$,

b) $\mu_{t,\epsilon} \le C_\epsilon \mathfrak{m}$ for every $t \in [0,1]$ and some constant $C_\epsilon > 0$,

c) $l[\mu_{\cdot,\epsilon}] \le \mathrm{d}(x,y) + \mathrm{err}(\varepsilon)$ for some $\mathrm{err}(\epsilon) > 0$ such that $\lim_{\epsilon\downarrow 0}\mathrm{err}(\epsilon) = 0$.

In what follows we shall say that a space $(X,\mathrm{d},\mathfrak{m})$ is $\mathfrak{m}$-a.e. locally doubling provided for $\mathfrak{m}$-a.e. $x \in X$ there exist $C,R > 0$ such that

$$\mathfrak{m}(B_{2r}(y)) \le C\mathfrak{m}(B_r(y)), \qquad \forall y \in B_R(x),\ r \le R. \tag{3.25}$$

We recall that on a doubling space $\mathfrak{m}$-a.e. point is a Lebesgue point of a given L^1_{loc} function. As the definition of Lebesgue point is local in nature, we see that the same property holds in $\mathfrak{m}$-a.e. locally doubling spaces in the above sense.

The concepts of good and $\mathfrak{m}$-a.e. locally doubling spaces are linked to the Sobolev-to-Lipschitz property via the following simple result:

Proposition 3.29. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathfrak{m}$-a.e. locally doubling and good space. Then it has the Sobolev-to-Lipschitz property.*

Proof. Let $f \in W^{1,2}(X)$ with ess sup $|\mathrm{D}f| < \infty$ and $B \subset X$ the set of Lebesgue points of f. Since $(X, \mathrm{d}, \mathfrak{m})$ is $\mathfrak{m}$-a.e. locally doubling we know that $\mathfrak{m}(X \setminus B) = 0$. Let $A \subset X$ be the set given in the definition of good space and pick $x, y \in A \cap B$. Then we know that there exists a family of curves $t \mapsto \mu_{t,\epsilon}$ as in Definition 3.28. Up to a reparametrization we can assume that such curves have constant speed, so that in particular such speeds are in $L^2(0,1)$ and we can apply the superposition principle in Theorem 3.3 to find plans π_ϵ such that for every $\epsilon > 0$ we have $(e_t)_\sharp(\pi_\epsilon) = \mu_{t,\epsilon}$ for every $t \in [0,1]$ and $\int |\dot{\gamma}_t|^2 \, \mathrm{d}\pi_\epsilon(\gamma) = |\dot{\mu}_{t,\epsilon}|^2$ for a.e. $t \in [0,1]$. These two facts together with the bound $\mu_{t,\epsilon} \le C_\epsilon \mathfrak{m}$ grant that π_ϵ is a test plan. Therefore we have:

$$\begin{aligned}
\int |f(\gamma_0) - f(\gamma_1)| \, \mathrm{d}\pi_\epsilon(\gamma) &\le \int_0^1 \int |\mathrm{D}f|_X(\gamma_t) |\dot{\gamma}_t| \mathrm{d}\pi_\epsilon(\gamma) \, \mathrm{d}t \\
&\le (\text{ess sup}|\mathrm{D}f|) \int_0^1 \int |\dot{\gamma}_t| \mathrm{d}\pi_\epsilon \, \mathrm{d}t \\
&\le (\text{ess sup}|\mathrm{D}f|) \sqrt{\int_0^1 \int |\dot{\gamma}_t|^2 \mathrm{d}\pi_\epsilon \, \mathrm{d}t} \\
&= (\text{ess sup}|\mathrm{D}f|) \sqrt{\int_0^1 |\dot{\mu}_{t,\epsilon}|^2 \, \mathrm{d}t} \\
&= (\text{ess sup}|\mathrm{D}f|) l[\mu_{\cdot,\epsilon}],
\end{aligned}$$

where in the last step we used the fact that $t \mapsto \mu_{t,\epsilon}$ has constant speed.

Letting ϵ go to 0 and using the fact that x, y are Lebesgue points we get

$$\begin{aligned}
|f(y) - f(x)| &= \lim_{\epsilon \to 0} \left| \int f \, \mathrm{d}\mu_{1,\epsilon} - \int f \, \mathrm{d}\mu_{0,\epsilon} \right| \\
&= \lim_{\epsilon \to 0} \left| \int f(\gamma_1) - f(\gamma_0) \, \mathrm{d}\pi_\epsilon(\gamma) \right| \\
&\le (\text{ess sup}|\mathrm{D}f|) \mathrm{d}(x, y).
\end{aligned}$$

This proves that the restriction of f to $A \cap B$ is Lipschitz with Lipschitz constant bounded by ess sup$|\mathrm{D}f|$. Since $A \cap B$ has full $\mathfrak{m}$-measure, this concludes the proof. $\square$

We now turn to the main result of the section:

Theorem 3.30. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a doubling $\mathrm{RCD}(K, \infty)$ space, $I \subset \mathbb{R}$ a closed, possibly unbounded interval and $w_{\mathrm{d}}, w_{\mathfrak{m}} : I \to \mathbb{R}$ a couple of warping functions. Assume that $w_{\mathfrak{m}}$ is strictly positive in the interior of I.*

Then the warped product $(X_w, \mathrm{d}_w, \mathfrak{m}_w)$ has the Sobolev to Lipschitz property.

Proof. It is trivial to check that the cartesian product of two doubling spaces is still doubling. It follows that if $(t, x) \in X_w$ is such that $w_{\mathfrak{m}}(t) > 0$, then property (3.25) is satisfied for a sufficiently small R and some constant C. Since by definition of $\mathfrak{m}_w$ we know that $\mathfrak{m}_w$-a.e. $(t, x) \in X_w$ is such that $w_{\mathfrak{m}}(t) > 0$, we deduce that $(X_w, \mathrm{d}_w, \mathfrak{m}_w)$ is $\mathfrak{m}_w$-a.e. locally doubling. Hence to conclude it is sufficient to show that it is a good space. We shall divide the proof of this fact in two cases.

Step 1. We assume that $w_{\mathfrak{m}}$ is strictly positive on the whole I.

Let $(t_0, x_0), (t_1, x_1) \in X_w$ be with $t_0 < t_1$, $\epsilon > 0$ and $\gamma = (\gamma^{\mathbb{R}}, \gamma^X)$ a curve joining (t_0, x_0) to (t_1, x_1) with $l_w(\gamma) \leq \mathrm{d}_w((t_0, x_0), (t_1, x_1)) + \epsilon$. Then the curve $\gamma^{\mathbb{R}}$ has image contained in $J := I \cap [t_0 - l_w(\gamma) - \epsilon, t_1 + l_w(\gamma) + \epsilon]$.

The function w_{d} is strictly positive on J (because $\{w_{\mathrm{d}} = 0\} \subset \{w_{\mathfrak{m}} = 0\} = \emptyset$) and continuous. Hence $\log(w_{\mathrm{d}})$ is uniformly continuous on J and we can find $\delta \in (0, \epsilon \wedge 1)$ so that

$$\left|\frac{w_{\mathrm{d}}(t)}{w_{\mathrm{d}}(s)}\right|^2 \leq 1 + \epsilon \qquad \text{for every } t, s \in J \text{ with } |t - s| \leq \delta. \tag{3.26}$$

Now let N be the integer part of $\frac{2}{\delta} + 1$, let $0 = t_0 < \ldots < t_N = 1$ be such that $|t_{i+1} - t_i| \leq \frac{\delta}{2}$ for every $i = 0, \ldots, N-1$, and define the measures

$$\mu_i := \frac{1_{B_{\delta^2/4}(\gamma_{t_i})}}{\mathfrak{m}_w(B_{\delta^2/4}(\gamma_{t_i}))}\mathfrak{m}_w \in \mathcal{P}_2(X_w), \qquad\qquad i = 0, \ldots, N.$$

For every $i = 1, \ldots, N-1$ consider the constant warping functions $w_{i,\mathrm{d}}(t) := w_{\mathrm{d}}(t_i)$ and $w_{i,\mathfrak{m}}(t) := w_{\mathfrak{m}}(t_i)$ and the corresponding warped product spaces, which we shall denote as $(X_i, \mathrm{d}_i, \mathfrak{m}_i)$, of X and $\mathbb{R}$. These warped products are in fact cartesian products of a rescaled version of X, which is an $RCD(K', \infty)$ space for some K' - see [40], and $\mathbb{R}$ and therefore they are $RCD(K', \infty)$. Consider μ_i, μ_{i+1} as measures on X_i and let $[t_i, t_{i+1}] \ni t \mapsto \mu_{i,t}$ be the $W_2^{\mathrm{d}_i}$-geodesic connecting them and notice that $\mu_{i,t}$ has support in the strip $[t_i - \delta/2, t_{i+1} + \delta/2] \times X$. Since $w_{\mathfrak{m}}$ is bounded from above on J, we see that μ_i, μ_{i+1} have density w.r.t. $\mathfrak{m}_i$ bounded from above. Therefore by (3.24) we deduce that

$$\mu_{i,t} \leq C_i \mathfrak{m}_i, \qquad \forall t \in [t_i, t_{i+1}], \tag{3.27}$$

for some constant C_i.

The bound (3.26) easily yields that for $(t,x)\neq(s,y)\in[t_i-\delta/2,t_{i+1}+\delta/2]\times X$ we have

$$\frac{1}{1+\epsilon}\leq\frac{\mathrm{d}_w^2\big((t,x),(s,y)\big)}{\mathrm{d}_i^2\big((t,x),(s,y)\big)}\leq 1+\epsilon.$$

Then the analogous inequality is true for metric speeds, for $\mathcal{W}_2$-distances between probability measure concentrated on $[t_i-\delta/2,t_{i+1}+\delta/2]\times X$ and metric speeds of curves of measures. Thus we have

$$\begin{aligned} l_w(\mu_{i,\cdot})=\int_{t_i}^{t_{i+1}}|\dot\mu_{i,t}|_w\,\mathrm{d}t\leq\sqrt{1+\epsilon}\int_{t_i}^{t_{i+1}}|\dot\mu_{i,t}|_i\,\mathrm{d}t& \\ =\sqrt{1+\epsilon}\,\mathcal{W}_2^{\mathrm{d}_i}(\mu_i,\mu_{i+1})\leq(1+\epsilon)\mathcal{W}_2^{\mathrm{d}_w}(\mu_i,\mu_{i+1}).& \end{aligned}\tag{3.28}$$

We then define the curve $[0,1]\ni t\mapsto\mu_t\in\mathcal{P}(X_w)$ as $\mu_t:=\mu_{i,t}$ if $t\in[t_i,t_{i+1}]$ which is, by the above discussion, absolutely continuous. It clearly satisfies condition (a) in Definition 3.28. Property (b) follows directly from (3.27) and the fact that $w_{\mathfrak{m}}$ is bounded from below on J. To prove that it has property (c), notice that by construction we have $\mathcal{W}_2^{\mathrm{d}_w}(\mu_i,\mu_{i+1})\leq\mathrm{d}_w(\gamma_{t_i},\gamma_{t_{i+1}})+\delta^2/2$, therefore from (4.7) we deduce

$$\begin{aligned} l_w(\mu_t)=\sum_{i=0}^{N-1}l_w(\mu_{i,t})\leq(1+\epsilon)\sum_{i=0}^{N-1}\big(\mathrm{d}_w(\gamma_{t_i},\gamma_{t_{i+1}})+\delta^2/2\big)& \\ \leq(1+\epsilon)\Big(\frac{2}{\delta}+1\Big)\frac{\delta^2}{2}+(1+\epsilon)\sum_{i=0}^{N-1}\mathrm{d}_w(\gamma_{t_i},\gamma_{t_{i+1}})& \\ \leq 2\epsilon(1+\epsilon)+(1+\epsilon)l_w(\gamma)& \\ \leq(1+\epsilon)\Big(2\epsilon+\mathrm{d}_w\big((t_0,x_0),(t_1,x_1)\big)\Big),& \end{aligned}$$

which is our claim.

Step 2. <u>We drop the positivity assumption of $w_{\mathfrak{m}}$ on the extrema of I.</u>

We define $A\subset X_w$ as the product of the interior of I and X. Notice that A has full $\mathfrak{m}_w$ measure. Let $(I_\delta)_{\delta>0}$ be a family of bounded closed intervals such that $I_\delta\supset I_{\delta'}$ for $\delta<\delta'$ whose union is the interior of I, pick $(t_0,x_0),(t_1,x_1)\in A$ and let $\delta_0>0$ so that $t_0,t_1\in I_{\delta_0}$.

For every $\delta\in(0,\delta_0)$ consider the warped product of I_δ and X via the warping functions $w_{\mathrm{d}},w_{\mathfrak{m}}$, which we shall denote by $(X_\delta,\mathrm{d}_\delta,\mathfrak{m}_\delta)$. From the continuity of w_{d} it is easy to check that a curve with values in X_δ is d_δ-absolutely continuous if and only if it is d_w-absolutely continuous and in this case the two metric speeds agree. Then the analogous statement is valid for curves of probability measures and their $\mathcal{W}_2$-speeds in the two spaces.

Now observe that the inequality $\mathrm{d}_w \leq \mathrm{d}_\delta$ is obvious and that given a curve $t \mapsto (\gamma_t^{\mathbb{R}}, \gamma_t^X) \in X_w$ the curves $t \mapsto (\delta \vee \gamma_t^{\mathbb{R}}, \gamma_t^X) \in X_\delta$ have length converging to that of the original curve as $\delta \downarrow 0$. It follows that

$$\lim_{\delta \to 0} \mathrm{d}_\delta\big((t_0, x_0), (t_1, x_1)\big) = \mathrm{d}_w\big((t_0, x_0), (t_1, x_1)\big).$$

The conclusion then follows by applying the previous step in the space X_δ, passing to the limit as $\delta \downarrow 0$ and using a diagonalization argument to produce the desired family of curves of measures in X_w. $\square$

Chapter 4

Independence on p of weak upper gradients on RCD spaces

Abstract

In this chapter, we study p-weak gradients on $\mathrm{RCD}(K,\infty)$ metric measure spaces and prove that they all coincide for $p>1$. On proper spaces, our arguments also cover the extremal situation of BV functions.

Résumé

Dans ce chapitre, nous étudions p-gradients faibles dans les espaces métriques mesurés. Sous une condition de courbure-dimension $\mathrm{RCD}(K,\infty)$, nous montrons l'identification des p-gradients faibles. Dans les espaces propres, nos arguments couvrent également la situation des fonctions à variation bornée.

The results in this chapter are contained in [26].

4.1 Introduction

There is a large literature concerning the definition of the Sobolev space $W^{1,p}(X,\mathrm{d},\mathfrak{m})$ of real valued functions defined on a metric measure space $(X,\mathrm{d},\mathfrak{m})$, we refer to [30] and [6] for historical comments and a presentation of the various - mostly equivalent - approaches.

The definition of space $W^{1,p}(X,\mathrm{d},\mathfrak{m})$ comes with the definition of an object playing the role of the modulus of the distributional differential. More precisely, for $f\in$

$W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ it is well defined a non-negative function $|\mathrm{D}f|_p \in L^p(X, \mathfrak{m})$, called minimal p-weak upper gradient, which, if $(X, \mathrm{d}, \mathfrak{m})$ is a smooth space, coincides $\mathfrak{m}$-a.e. with the modulus of the distributional differential of f.

A key difference between the smooth and non-smooth case is that in the latter the minimal p-weak upper gradient may depend on p: say for simplicity that $\mathfrak{m}(X) = 1$, then for $p < q \in (1, \infty)$ and $f \in W^{1,q}(X)$ one always has $f \in W^{1,p}(X)$ but in general only the inequality

$$|\mathrm{D}f|_p \leq |\mathrm{D}f|_q, \qquad \mathfrak{m}- \text{ a.e.}, \tag{4.1}$$

holds. The inequality above can be strict even on doubling spaces, see [36] for an example and more details on the issue.

Worse than this, one might have

$$\text{a function } f \in W^{1,p}(X) \text{ with } f, |\mathrm{D}f|_p \in L^q(X) \text{ such that } f \notin W^{1,q}(X), \tag{4.2}$$

see [6] for an example proposed by Koskela.

To have a p-weak upper gradients independent on p is a regularity property of the metric measure space in question. For instance, as a consequence of the analysis done in [20] one has that on doubling space supporting a 1-1 weak local Poincaré inequality, equality always holds in (4.1). In particular, this applies to $\mathsf{CD}(K, N)$ spaces with $N < \infty$.

In this note we show that on RCD(K, ∞) spaces not only (4.1) holds with equality, but also that the situation in (4.2) never occurs. The argument is based on some regularization properties of the heat flow proved in [39] and on the density in energy of Lipschitz functions in Sobolev spaces established in [6].

At least in the case of proper RCD(K, ∞) spaces, this identification extends to BV functions. The problem in non-proper spaces is the lack of an approximation result of BV functions with Lipschitz ones.

This result, beside its intrinsic usefulness in Sobolev calculus, has also the pleasant conceptual effect of somehow relieving the definition of RCD(K, ∞) spaces from the dependence on the particular Sobolev exponent $p = 2$. Recall indeed that one of the equivalent definitions of RCD(K, ∞) space is that of a CD(K, ∞) space such that $W^{1,2}(X)$ is Hilbert or equivalently such that

$$|\mathrm{D}(f+g)|_2^2 + |\mathrm{D}(f-g)|_2^2 = 2\big(|\mathrm{D}f|_2^2 + |\mathrm{D}g|_2^2\big), \quad \mathfrak{m}\text{-a.e.} \qquad \forall f, g \in W^{1,2}(X).$$

As a consequence of our result, a posteriori one could replace the minimal 2-weak upper gradients with p-weak upper gradients in the above.

4.2 Preliminaries

4.2.1 Sobolev classes

We assume the reader familiar with the basic concepts of analysis in metric measure spaces and recall here the definition of Sobolev class $S^p(X)$. We fix a complete and separable space $(X, \mathrm{d}, \mathfrak{m})$ such that $\mathfrak{m}$ is a non-negative Borel measure finite on bounded sets.

Definition 4.1 (Test plans)**.** Let π be a Borel probability measure on $C([0,1], X)$. We say that π has bounded compression provided there exists $C = C(\pi) > 0$ such that

$$(e_t)_\sharp \pi \le C\mathfrak{m}, \qquad \forall t \in [0,1],$$

where $e_t : C([0,1], X) \mapsto X$ is the evaluation map defined by $e_t(\gamma) := \gamma_t$ for every $\gamma \in C([0,1], X)$.

For $q \in (1, \infty)$ we say that π is a q-test plan if it has bounded compression, is concentrated on $AC^q([0,1], X)$ and

$$\int_0^1 \int |\dot\gamma_t|^q \, \mathrm{d}\pi(\gamma) \, \mathrm{d}t < +\infty.$$

The notion of Sobolev function is then introduced by duality with test plans.

Definition 4.2 (Sobolev classes)**.** Let $p \in (1, \infty)$. The space $S^p(X)$ is the space of all Borel functions $f : X \mapsto \mathbb{R}$ for which there exists a non-negative function $G \in L^p(X)$ such that for any q-test plan π the inequality

$$\int |f(\gamma_1) - f(\gamma_0)| \, \mathrm{d}\pi(\gamma) \le \int \int_0^1 G(\gamma_s)|\dot\gamma_s| \, \mathrm{d}s \, \mathrm{d}\pi(\gamma)$$

holds, where $\frac{1}{p} + \frac{1}{q} = 1$. Any such G is called p-weak upper gradient.

It is immediate to see that for $f \in S^p(X)$ there is a unique p-weak upper gradient of minimal L^p-norm: we shall call such G minimal p-weak upper gradient and denote it by $|\mathrm{D}f|_p$.

Basic important properties of minimal weak upper gradients are the locality, i.e.:

$$|\mathrm{D}f|_p = |\mathrm{D}g|_p, \quad \mathfrak{m}\text{-a.e. on } \{f = g\}, \qquad \forall f, g \in S^p(X),$$

and the lower semicontinuity, i.e.

$$\left.\begin{array}{l}(f_n) \subset S^p(X),\\ \sup_n \||\mathrm{D}f|_p\|_{L^p} < \infty,\\ f_n \to f \quad \mathfrak{m}\text{-a.e.}\end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} f \in S^p(X) \text{ and}\\ \text{for any weak limit } G \text{ of } (|\mathrm{D}f_n|_p) \text{ in } L^p(X)\\ \text{it holds } |\mathrm{D}f|_p \le G \qquad \mathfrak{m}\text{-a.e.}\end{array}\right.$$

The Sobolev space $W^{1,p}(X)$ is defined as $W^{1,p}(X) := S^p \cap L^p(X)$ endowed with the norm

$$\|f\|^p_{W^{1,p}(X)} := \|f\|^p_{L^p(X)} + \||\mathrm{D}f|_p\|^p_{L^p(X)}.$$

By $\mathrm{Lip}\, X$ we denote the space of Lipschitz functions on X and for $f \in \mathrm{Lip}\, X$ the local Lipschitz constant $\mathrm{lip}(f) : X \mapsto [0, \infty)$ is defined as

$$\mathrm{lip}(f)(x) := \varlimsup_{y \to x} \frac{|f(y) - f(x)|}{\mathrm{d}(x, y)},$$

if x is not isolated, 0 otherwise.

In [6] the following approximation property has been proved:

Proposition 4.3 (Density in energy of Lipschitz functions)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a complete separable metric space with $\mathfrak{m}$ being Borel non-negative and assigning finite mass to bounded sets. Let $p \in (1, \infty)$ and $f \in W^{1,p}(X)$.*

Then there exists a sequence $(f_n) \subset W^{1,p} \cap \mathrm{Lip}\, X$ of functions with bounded support converging to f in $L^p(X)$ and such that $\mathrm{lip}(f_n) \to |\mathrm{D}f|_p$ in $L^p(X)$ as $n \to \infty$.

We conclude this introduction noticing that the locality property of p-weak upper gradients allows for a natural definition of the space of locally Sobolev functions. By $L^p_{\mathrm{loc}}(X)$ we shall intend the space of Borel functions $G : X \mapsto \mathbb{R}$ whose p-power is integrable on every bounded set.

Definition 4.4 (The spaces $S^p_{\mathrm{loc}}(X)$)**.** We say that $f \in S^p_{\mathrm{loc}}(X)$ provided for any Lipschitz function with bounded support χ we have $\chi f \in S^p(X)$. In this case we define $|\mathrm{D}f|_p \in L^p_{\mathrm{loc}}(X)$ as

$$|\mathrm{D}f|_p := |\mathrm{D}(\chi f)|_p, \qquad \mathfrak{m}\text{-a.e. on } \{\chi = 1\},$$

for every χ as before.

The role of the locality of the p-weak upper gradient is to ensure that the definition of $|\mathrm{D}f|_p$ is well posed. Also, it is not hard to check that $S^p(X) \subset S^p_{\mathrm{loc}}(X)$ and that a function $f \in S^p_{\mathrm{loc}}(X)$ belongs to $S^p(X)$ if and only if $|\mathrm{D}f|_p \in L^p(X)$.

For $p_1 < p_2 \in (1,\infty)$ and $q_1 > q_2 \in (1,\infty)$ such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$, the fact that the class of q_1-test plans is contained in the one of q_2-test plans grants that

$$S^{p_2}_{\rm loc}(X) \subset S^{p_1}_{\rm loc}(X) \qquad \text{and} \qquad |{\rm D}f|_{p_1} \le |{\rm D}f|_{p_2} \quad \mathfrak{m}\text{-a.e.} \qquad \forall f \in S^{p_2}_{\rm loc}(X). \tag{4.3}$$

4.2.2 Heat flow on $\mathrm{RCD}(K,\infty)$ space

In order to keep this preliminary part as short as possible, we shall assume the reader familiar with the definition of $\mathrm{RCD}(K,\infty)$ spaces and focus only on those properties they have which are relevant for our discussion. We refer to [9], [4] and [39] for the throughout discussion.

From now on we shall assume that $(X,{\rm d},\mathfrak{m})$ is a $\mathrm{RCD}(K,\infty)$ space for some $K\in\mathbb{R}$ and that the support of $\mathfrak{m}$ is the whole X. Recall that in particular we have $\mathfrak{m}(B)<\infty$ for any bounded Borel set $B\subset X$.

In such space the 2-Energy functional $\mathbb{E} : L^2(X) \mapsto [0,\infty]$ defined as

$$\mathbb{E}_2(f) := \begin{cases} \frac{1}{2}\int_X |{\rm D}f|_2^2 \,{\rm d}\mathfrak{m}, & \text{if } f \in W^{1,2}(X), \\ +\infty, & \text{otherwise,} \end{cases}$$

is a strongly local and regular Dirichlet form. We shall denote by $({\rm H}_t)$ the associated linear semigroup. Then it can be seen that for every $f\in L^2(X)$ and $p\in[1,\infty)$ we have

$$\|{\rm H}_t(f)\|_{L^p(X)} \le \|f\|_{L^p(X)}, \qquad \forall t\ge 0,$$

and thus $({\rm H}_t)$ can, and will, be extended to a linear non-expanding semigroup on $L^p(X)$.

On the other hand, there exists a unique EVI_K-gradient flow of the relative entropy functional on $(\mathcal{P}_2,\mathcal{W}_2)$ which we shall denote by $(\mathcal{H}_t)$ and provides a one parameter semigroup of continuous linear operators on $(\mathcal{P}_2,\mathcal{W}_2)$, see [8, 9] and [4].

The non-trivial link between $({\rm H}_t)$ and $(\mathcal{H}_t)$ is the fact that

$$\begin{aligned} &\text{for } \mu\in\mathcal{P}(X) \text{ such that } \mu = f\mathfrak{m} \text{ for some } f\in L^2(X) \\ &\text{we have } \mathcal{H}_t(\mu) = {\rm H}_t(f)\mathfrak{m} \text{ for every } t\ge 0, \end{aligned}$$

and from the fact that $({\rm H}_t)$ is self adjoint one can verify that for any $p\in[1,\infty)$ and every $t\ge0$ it holds

$${\rm H}_t(f)(x) = \int f\,{\rm d}\mathcal{H}_t(\delta_x), \qquad \mathfrak{m}-\ \text{a.e.. } x \qquad \forall f\in L^p(X). \tag{4.4}$$

Moreover, for $f \in L^\infty(X)$ and $t > 0$ the formula

$$\tilde{\mathrm{H}}_t(f)(x) := \int f \,\mathrm{d}\mathcal{H}_t(\delta_x),$$

is well defined for any $x \in X$ producing a pointwise version of the heat flow for which the $L^\infty \mapsto \mathrm{Lip}$ regularization holds:

$$\mathrm{Lip}\,\tilde{\mathrm{H}}_t(f) \leq \frac{1}{\sqrt{2I_{2K}(t)}}\|f\|_{L^\infty(X)}, \qquad \forall t > 0, \tag{4.5}$$

where $I_{2K}(t) := \int_0^t e^{2Ks}\,\mathrm{d}s$.

Th crucial regularization property of the heat flow that we shall use to identify p-weak gradients is the following version of the Bakry-Émery contraction rate, proved in [**?**]:

$$\mathrm{lip}(\tilde{\mathrm{H}}_t(f)) \leq e^{-Kt}\tilde{\mathrm{H}}_t(\mathrm{lip}(f)) \qquad \text{pointwise on } X, \tag{4.6}$$

valid for every Lipschitz function f with bounded support and every $t \geq 0$.

We conclude recalling another useful regularity property of $\mathrm{RCD}(K,\infty)$ spaces, this one concerning displacement interpolation of measures, see [38] for a proof:

Proposition 4.5. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(K,\infty)$ space and μ, ν two Borel probability measures with bounded support and such that $\mu, \nu \leq C\mathfrak{m}$ for some $C > 0$.*

Then there exists a Borel probability measure π on $C([0,1], X)$ such that $(e_t)_\sharp\pi \leq C'\mathfrak{m}$ for every $t \in [0,1]$ for some $C' > 0$ and for which the inequality

$$\mathrm{Lip}\,\gamma \leq \sup_{\substack{x\in\mathrm{supp}(\mu)\\ y\in\mathrm{supp}(\nu)}} \mathrm{d}(x,y),$$

holds for every γ in the support of π.

4.3 Proof of the main result

The identification of p-weak gradients will come via a study of the regularization properties of the heat flow.

Proposition 4.6. *Let $p \in (1,\infty)$, $f \in W^{1,p}(X)$ and $t \geq 0$. Then $\mathrm{H}_t(f) \in W^{1,p}(X)$ and*

$$|\mathrm{DH}_t f|_p^p \leq e^{-pKt}\mathrm{H}_t(|\mathrm{D}f|_p^p), \qquad \mathfrak{m}-\quad a.e..$$

Proof. The fact that $\mathrm{H}_t(f) \in L^p(X)$ follows from the fact that $f \in L^p(X)$. By Proposition 4.3 we can find a sequence $(f_n) \subset W^{1,p} \cap \mathrm{Lip}\,X$ converging to f in $L^p(X)$ and such that $\mathrm{lip}(f_n) \to |\mathrm{D}f|_p$ in $L^p(X)$. By the property (4.6) we know that

$$\mathrm{lip}(\tilde{\mathrm{H}}_t(f_n))^p \le e^{-pKt}\tilde{\mathrm{H}}_t(\mathrm{lip}(f_n)^p), \qquad \text{pointwise on } X. \tag{4.7}$$

The continuity in $L^1(X)$ of the heat flow grants that

$$\tilde{\mathrm{H}}_t(|\mathrm{lip}(f_n)|^p) \to \tilde{\mathrm{H}}_t(|\mathrm{D}f|_p^p), \qquad \text{in } L^1(X), \tag{4.8}$$

so that in particular (4.7) grants that the sequence $(\mathrm{lip}(\tilde{\mathrm{H}}_t(f_n)))$ is bounded in $L^p(X)$. Therefore also $(|\mathrm{D}\tilde{\mathrm{H}}_t(f_n)|_p)$ is bounded in $L^p(X)$ and up to pass to a subsequence, not relabeled, we can assume that it weakly converges to some $G \in L^p(X)$. By (4.7) and (4.8) we have $G \le \tilde{\mathrm{H}}_t(|\mathrm{D}f|_p^p)$ $\mathfrak{m}$-a.e. while the lower semicontinuity of p-weak upper gradients ensures that $\mathrm{H}_t(f) \in S^p(X)$ with $|\mathrm{D}f|_p \le G$ $\mathfrak{m}$-a.e. and the thesis follows. $\square$

Proposition 4.7. *Let $p \in (1,\infty)$, $f \in W^{1,p}(X)$ such that $f, |\mathrm{D}f|_p \in L^\infty(X)$ and $t > 0$. Then $\tilde{\mathrm{H}}_t(f)$ is Lipschitz and*

$$\mathrm{lip}(\tilde{\mathrm{H}}_t(f)) \le e^{-Kt}\sqrt[p]{\tilde{\mathrm{H}}_t(|\mathrm{D}f|_p^p)}, \qquad \textit{pointwise on } X.$$

Proof. The fact that $\tilde{\mathrm{H}}_t(f)$ is Lipschitz follows from (4.5). To prove the thesis, pick $x, y \in X$, $r > 0$, consider the measures $\mu_{0,r} := \mathfrak{m}(B_r(x))^{-1}\mathfrak{m}|_{B_r(x)}$, $\mu_{1,r} := \mathfrak{m}(B_r(y))^{-1}\mathfrak{m}|_{B_r(y)}$ and let π be given by Proposition 4.5. Then we know that $(e_s)_\sharp\pi \le C\mathfrak{m}$ for some $C > 0$ and every $s \in [0,1]$ and that $|\dot\gamma_s| \le \mathrm{d}(x,y) + 2r$ for π-a.e. γ and a.e. $s \in [0,1]$. In particular, π is a q-test plan, where $\frac{1}{p} + \frac{1}{q} = 1$, and since $\mathrm{H}_t(f) \in W^{1,p}(X)$ we know that

$$\begin{aligned}
\Big|\int \tilde{\mathrm{H}}_t(f)\,\mathrm{d}(\mu_{1,r} - \mu_{0,r})\Big| &\le \int |\tilde{\mathrm{H}}_t(f)(\gamma_1) - \tilde{\mathrm{H}}_t(f)(\gamma_0)|\,\mathrm{d}\pi(\gamma)\\
&\le \iint_0^1 |\mathrm{D}\tilde{\mathrm{H}}_t(f)|_p(\gamma_s)|\dot\gamma_s|\,\mathrm{d}s\,\mathrm{d}\pi(\gamma)\\
&\le (\mathrm{d}(x,y) + 2r)\sqrt[p]{\iint_0^1 |\mathrm{D}\tilde{\mathrm{H}}_t(f)|_p^p(\gamma_s)\,\mathrm{d}s\,\mathrm{d}\pi(\gamma)}\\
&\le (\mathrm{d}(x,y) + 2r)e^{-Kt}\sqrt[p]{\int\int_0^1 \tilde{\mathrm{H}}_t(|\mathrm{D}f|_p^p)(\gamma_s)\,\mathrm{d}s\,\mathrm{d}\pi(\gamma)},
\end{aligned}$$

having used Proposition 4.6 in the last step. Noticing that $\mathrm{d}(x,\gamma_s) \le \mathrm{d}(x,y) + 3r$ for π-a.e. γ and every $s \in [0,1]$ we deduce that

$$\Big|\int \tilde{\mathrm{H}}_t(f)\,\mathrm{d}(\mu_{1,r} - \mu_{0,r})\Big| \le (\mathrm{d}(x,y) + 2r)e^{-Kt}\sqrt[p]{\sup_{B_{\mathrm{d}(x,y)+3r}} \tilde{\mathrm{H}}_t(|\mathrm{D}f|_p^p)},$$

and letting $r \downarrow 0$ and using the continuity of $\tilde{\mathrm{H}}_t(f)$ we deduce that

$$\frac{|\tilde{\mathrm{H}}_t(f)(y) - \tilde{\mathrm{H}}_t(f)|}{\mathrm{d}(x,y)} \leq e^{-Kt} \sqrt[p]{\sup_{B_{\mathrm{d}(x,y)+\varepsilon}} \tilde{\mathrm{H}}_t(|\mathrm{D}f|_p^p)}, \qquad \forall \varepsilon > 0.$$

Letting $y \to x$ using the continuity of $\tilde{\mathrm{H}}_t(|\mathrm{D}f|_p^p)$ (which follows from the hypothesis $|\mathrm{D}f|_p \in L^\infty(X)$ and (4.5)) and the arbitrariness of $\varepsilon > 0$ we conclude. □

Proposition 4.8. *Let $p, q \in (1,\infty)$ and $f \in \mathrm{Lip}\, X$. Then*

$$|\mathrm{D}f|_q = |\mathrm{D}f|_p, \qquad \mathfrak{m} - \ a.e..$$

Proof. Assume $p < q$. Then we already know by (4.3) that $|\mathrm{D}f|_p \leq |\mathrm{D}f|_q$ $\mathfrak{m}$-a.e.. Notice that by the locality property of the weak upper gradients it is not restrictive to assume that f has bounded support, so that in particular $f \in L^\infty \cap W^{1,p}(X)$. Let $t > 0$ and apply Proposition 4.7 to deduce that $\tilde{\mathrm{H}}_t(f) \in \mathrm{Lip}\, X$ with

$$\mathrm{lip}(\tilde{\mathrm{H}}_t(f))^q \leq e^{-qKt}\tilde{\mathrm{H}}_t(|\mathrm{D}f|_p^p)^{\frac{q}{p}} \leq e^{-qKt}\tilde{\mathrm{H}}_t(|\mathrm{D}f|_p^q), \qquad \text{pointwise},$$

having used Jensen's inequality and formula (4.4) in the last step and the fact that $|\mathrm{D}f|_p^q \in L^1(X)$, which follows from the fact that f is Lipschitz bounded support. Since $|\mathrm{D}\tilde{\mathrm{H}}_t(f)|_q \leq \mathrm{lip}(\tilde{\mathrm{H}}_t(f))$ $\mathfrak{m}$-a.e., it follows that

$$\int |\mathrm{D}\tilde{\mathrm{H}}_t(f)|_q^q \,\mathrm{d}\mathfrak{m} \leq e^{-qKt} \int \tilde{\mathrm{H}}_t(|\mathrm{D}f|_p^q)\,\mathrm{d}\mathfrak{m}, \qquad \forall t > 0,$$

and letting $t \downarrow 0$ and using the lower semicontinuity of q-weak upper gradients we conclude that

$$\int |\mathrm{D}f|_q^q \,\mathrm{d}\mathfrak{m} \leq \int |\mathrm{D}f|_p^q \,\mathrm{d}\mathfrak{m},$$

which is sufficient to get the thesis. □

Theorem 4.9 (Identification of weak upper gradients)**.** *Let $p, q \in (1,\infty)$ and $f \in S^p_{\mathrm{loc}}(X)$ such that $|\mathrm{D}f|_p \in L^q_{\mathrm{loc}}(X)$. Then $f \in S^q_{\mathrm{loc}}(X)$ and*

$$|\mathrm{D}f|_q = |\mathrm{D}f|_p, \qquad \mathfrak{m} - \ a.e..$$

Proof. Assume that $p < q$ and notice that by (4.3) it is sufficient to prove that $|\mathrm{D}f|_p \geq |\mathrm{D}f|_q$ $\mathfrak{m}$-a.e.. Replacing if necessary f with $\max\{\min\{f,n\},-n\}$ and using the locality property of weak upper gradients and the arbitrariness of $n \in \mathbb{N}$ we can assume that $f \in L^\infty(X)$. Similarly, with a cut-off argument we reduce to the case in which f has bounded support and thus in particular $|\mathrm{D}f|_p \in L^p \cap L^q(X)$.

With these assumptions we have $f \in W^{1,p}(X)$ and thus for $t > 0$ Proposition 4.6 gives

$$|\mathrm{DH}_t f|_p \leq e^{-Kt} \sqrt[p]{\mathrm{H}_t(|\mathrm{D}f|_p^p)}, \qquad \mathfrak{m} - \text{ a.e..}$$

Moreover, the fact that f is bounded grants, by (4.5), that $\mathrm{H}_t(f)$ has a Lipschitz representative $\tilde{\mathrm{H}}_t(f)$ and thus Proposition 4.8 gives

$$|\mathrm{DH}_t f|_q \leq e^{-Kt} \sqrt[p]{\mathrm{H}_t(|\mathrm{D}f|_p^p)}, \qquad \mathfrak{m} - \text{ a.e..}$$

Using the assumption that $|\mathrm{D}f|_p \in L^q(X)$ and Jensen's inequality in formula (4.4) we deduce that $|\mathrm{DH}_t f|_q^q \leq e^{-qKt}\mathrm{H}_t(|\mathrm{D}f|_p^q)$ $\mathfrak{m}$-a.e. and thus

$$\int |\mathrm{DH}_t f|_q^q \,\mathrm{d}\mathfrak{m} \leq e^{-qKt} \int \mathrm{H}_t(|\mathrm{D}f|_p^q)\,\mathrm{d}\mathfrak{m}, \qquad \forall t > 0.$$

Letting $t \downarrow 0$ and using the lower semicontinuity of q-weak upper gradients we conclude that

$$\int |\mathrm{D}f|_q^q \,\mathrm{d}\mathfrak{m} \leq \int |\mathrm{D}f|_p^q \,\mathrm{d}\mathfrak{m},$$

which is sufficient to prove the thesis. □

Remark 4.10 (The case of *BV* functions). Recalling the notation and results of [2] about BV functions and denoting by $|\mathbf{D}f|$ the total variation measure of $f \in \mathrm{BV}(X)$, assume for a moment that $(X, \mathrm{d}, \mathfrak{m})$ is a *proper* (=bounded closed sets are compact) $\mathrm{RCD}(K, \infty)$ space. Then the very same arguments just used allow to prove that

$$\begin{aligned} &\text{if } f \in \mathrm{BV}(X) \text{ is such that } |\mathbf{D}f| \ll \mathfrak{m} \text{ with } \frac{\mathrm{d}|\mathbf{D}f|}{\mathrm{d}\mathfrak{m}} \in L^p_{\mathrm{loc}}(X) \text{ for some } p > 1, \\ &\text{then } f \in \mathcal{S}^p_{\mathrm{loc}}(X) \text{ and } |\mathrm{D}f|_p = \frac{\mathrm{d}|\mathbf{D}f|}{\mathrm{d}\mathfrak{m}} \ \mathfrak{m}\text{-a.e..} \end{aligned} \tag{4.9}$$

To see why, notice that the fact that (X, d) is proper and the definition of $\mathrm{BV}(X)$ ensures that for $f \in \mathrm{BV}(X)$ there is a sequence (f_n) of Lipschitz functions with bounded support such that $(f_n) \to f$ in $L^1(X)$ and $\mathrm{lip}(f_n)\mathfrak{m} \to |\mathbf{D}f|$ weakly in duality with $C_c(X)$. Hence arguing as for Proposition 4.6 one gets by approximation that

$$f \in \mathrm{BV}(X) \qquad \Rightarrow \qquad \mathrm{H}_t(f) \in \mathrm{BV}(X) \qquad |\mathbf{D}\mathrm{H}_t(f)| \leq e^{-Kt}\mathcal{H}_t(|\mathbf{D}f|). \tag{4.10}$$

Then, using the a priori estimates on the relative entropy of $\mathcal{H}_t(\mu)$ in terms of the mass of μ (see [4]) one obtains that for a sequence of non-negative measures (μ_n) weakly converging to some measure μ in duality with $C_b(X)$ and $t > 0$, the sequence $n \mapsto g_n := \frac{\mathrm{d}\mathcal{H}_t(\mu_n)}{\mathrm{d}\mathfrak{m}}$ converges to $g := \frac{\mathrm{d}\mathcal{H}_t(\mu)}{\mathrm{d}\mathfrak{m}}$ weakly in duality with $L^\infty(X)$. Therefore, for π as in the proof of Proposition 4.7 and $(f_n) \subset \mathrm{Lip}\, X$ converging to $f \in \mathrm{BV}(X)$ and so that

$\mathrm{lip}(f_n)\mathfrak{m} \to |\mathbf{D}f|$ weakly in duality with $C_b(X)$, we can pass to the limit in the inequality

$$\begin{aligned}\int |\mathrm{H}_t(f_n)(\gamma_1) - \mathrm{H}_t(f_n)(\gamma_0)|\,\mathrm{d}\pi(\gamma) &\le \iint_0^1 \mathrm{lip}(\mathrm{H}_t(f_n))(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi(\gamma)\\ &\le e^{-Kt}\iint_0^1 \mathrm{H}_t(\mathrm{lip}(f_n))(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi(\gamma),\end{aligned}$$

to deduce that

$$\int |\mathrm{H}_t(f)(\gamma_1) - \mathrm{H}_t(f)(\gamma_0)|\,\mathrm{d}\pi(\gamma) \le e^{-Kt}\int\int_0^1 \frac{\mathrm{d}\mathcal{H}_t(|\mathrm{D}f|_w)}{\mathrm{d}\mathfrak{m}}(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi(\gamma).$$

In particular, arguing as in the proof of Proposition 4.7 we get that

$$f \in BV \cap L^\infty(X),\ |\mathrm{D}f|_w \le C\mathfrak{m} \qquad \Rightarrow \qquad \mathrm{lip}(\mathrm{H}_t(f)) \le e^{-Kt}\mathcal{H}_t\Big(\frac{\mathrm{d}|\mathrm{D}f|_w}{\mathrm{d}\mathfrak{m}}\Big). \tag{4.11}$$

Then following the same lines of thought of Proposition 4.8 and Theorem 4.9 the claim (4.9) follows.

Notice also that from (4.10) and with a truncation and mollification argument we deduce that

> for $f \in \mathrm{BV}(X)$ with $|\mathbf{D}f| \ll \mathfrak{m}$ there is a sequence $(f_n) \subset \mathrm{Lip}\,X$ such that $f_n \to f$ and $\mathrm{lip}(f_n) \to \frac{\mathrm{d}|\mathbf{D}f|}{\mathrm{d}\mathfrak{m}}$ strongly in $L^1(X)$ as $n \to \infty$.

In particular, the three notions of space $W^{1,1}(X)$ discussed in [2] all coincide.

All this if the space is proper. It is very natural to expect that the same results hold even without this further assumption, but in the general case it seems necessary to define BV functions taking limits of locally Lipschitz functions, rather than Lipschitz ones (see the proof of Lemma 5.2 in [2]). The problem then consists in the fact that the property (4.6) is not available for locally Lipschitz functions with local Lipschitz constant in L^1.

■

Chapter 5

Ricci tensor on $\mathrm{RCD}^*(K, N)$ space

Abstract

In this chapter, we prove an improved Bochner inequality based on the curvature-dimension condition and give a definition of N-dimensional Ricci tensor on metric measure spaces.

Résumé

Dans ce chapitre, sous une condition de courbure-dimension $\mathrm{RCD}(K, \infty)$ et dans le cadre d'une théorie non-lisse de Bakry-Émery, nous obtenons une inégalité améliorée de Bochner et proposons une définition du N-tenseur de Ricci dans les espaces métriques mesurés.

The results in this chapter are contained in [29].

5.1 Introduction

Let M be a Riemannian manifold equipped with a metric tensor $\langle \cdot, \cdot \rangle : [TM]^2 \mapsto C^\infty(M)$. We have the Bochner formula

$$\Gamma_2(f) = \mathrm{Ricci}(\nabla f, \nabla f) + \|\mathrm{H}_f\|^2_{\mathrm{HS}}, \tag{5.1}$$

valid for any smooth function f, where $\|\mathrm{H}_f\|_{\mathrm{HS}}$ is the Hilbert-Schimidt norm of the Hessian $\mathrm{H}_f := \nabla \mathrm{d} f$ and the operator Γ_2 is defined by

$$\Gamma_2(f) := \frac{1}{2} L\Gamma(f, f) - \Gamma(f, Lf), \qquad \Gamma(f, f) := \frac{1}{2} L(f^2) - fLf$$

where $\Gamma(\cdot,\cdot) = \langle \nabla \cdot, \nabla \cdot \rangle$, and $L = \Delta$ is the Laplace-Beltrami operator.

In particular, if the Ricci curvature of M is bounded from below by K, i.e. $\mathrm{Ricci}(v,v)(x) \geq K|v|^2(x)$ for any $x \in M$ and $v \in T_xM$, and the dimension is bounded from above by $N \in [1,\infty]$, we have the Bochner inequality

$$\Gamma_2(f) \geq \frac{1}{N}(\Delta f)^2 + K\Gamma(f). \tag{5.2}$$

Conversely, it is not hard to show that the validity of (5.2) for any smooth function f implies that the manifold has lower Ricci curvature bound K and upper dimension bound N, or in short that it is a $\mathrm{CD}(K,N)$ manifold.

Being this characterization of the $\mathrm{CD}(K,N)$ condition only based on properties of L, one can take (5.2) as definition of what it means for a diffusion operator L to satisfy the $\mathrm{CD}(K,N)$ condition. This was the approach suggested by Bakry-Émery in [15], we refer to [16] for an overview on the subject.

Following this line of thought, one can wonder whether in this framework one can recover the definition of the Ricci curvature tensor and deduce from (5.2) that it is bounded from below by K. From (5.1) we see that a natural definition is

$$\mathrm{Ricci}(\nabla f, \nabla f) := \Gamma_2(f) - \|\mathrm{H}_f\|^2_{\mathrm{HS}}, \tag{5.3}$$

and it is clear that if $\mathrm{Ricci} \geq K$, then (5.2) holds with $N = \infty$. There are few things that need to be understood in order to make definition (5.3) rigorous and complete in the setting of diffusion operators:

1) If our only data is the diffusion operator L, how can we give a meaning to the Hessian term in (5.3)?

2) Can we deduce that the Ricci curvature defined as in (5.3) is actually bounded from below by K from the assumption (5.2)?

3) Can we include the upper bound on the dimension in the discussion? How the presence of N affects the definition of the Ricci curvature?

This last question has a well known answer: it turns out that the correct thing to do is to define, for every $N \geq 1$, a sort of 'N-dimensional' Ricci tensor as follows:

$$\mathrm{Ricci}_N(\nabla f, \nabla f) := \begin{cases} \Gamma_2(f) - \|\mathrm{H}_f\|^2_{\mathrm{HS}} - \frac{1}{N-n(x)}(\mathrm{tr}\mathrm{H}_f - Lf)^2, & \text{if } N > n, \\ \Gamma_2(f) - \|\mathrm{H}_f\|^2_{\mathrm{HS}} - \infty(\mathrm{tr}\mathrm{H}_f - Lf)^2, & \text{if } N = n, \\ -\infty, & \text{if } N < n, \end{cases} \tag{5.4}$$

where n is the dimension of the manifold (recall that on a weighted manifold in general we have $\mathrm{trH}_f \neq \Delta f$). It is then not hard to see that if $\mathrm{Ricci}_N \geq K$ then indeed (5.2) holds.

It is harder to understand how to go back and prove that $\mathrm{Ricci}_N \geq K$ starting from (5.2). A first step in this direction, which answers (1), is to notice that in the smooth setting the identity

$$2\mathrm{H}_f(\nabla g, \nabla h) = \Gamma(g, \Gamma(f,h)) + \Gamma(h, \Gamma(f,g)) - \Gamma(f, \Gamma(g,h))$$

for any smooth g, h characterizes the Hessian of f, so that the same identity can be used to define the Hessian starting from a diffusion operator only. The question is then whether one can prove any efficient bound on it starting from (5.2) only. The first results in this direction where obtained by Bakry in [13] and [14], and only recently Sturm [42] concluded the argument showing that (5.2) implies $\mathrm{Ricci}_N \geq K$. In Sturm's approach, the operator Ricci_N is not defined as in (5.4), but rather as

$$\mathrm{Ricci}_N(\nabla f, \nabla f)(x) := \inf_{g\,:\,\Gamma(f-g)(x)=0} \Gamma_2(g)(x) - \frac{(Lg)^2(x)}{N} \tag{5.5}$$

and it is part of his contribution the proof that this definition is equivalent to (5.4).

All this for smooth, albeit possibly abstract, structures. On the other hand, there is as of now a quite well established theory of (non-smooth) metric measure spaces satisfying a curvature-dimension condition: that of $\mathrm{RCD}^*(K,N)$ spaces introduced by Ambrosio-Gigli-Savaré (see [9] and [24]) as a refinement of the original intuitions of Lott-Sturm-Villani ([35] and [40, 41]) and Bacher-Sturm ([12]). In this setting, there is a very natural Laplacian and inequality (5.2) is known to be valid in the appropriate weak sense (see [9] and [21]) and one can therefore wonder if even in this low-regularity situation one can produce an effective notion of N-Ricci curvature. Part of the problem here is the a priori lack of vocabulary, so that for instance it is unclear what a vector field should be.

In the recent paper [23], Gigli builds a differential structure on metric measure spaces suitable to handle the objects we are discussing (see the preliminary section for some details). One of his results is to give a meaning to formula (5.3) on $\mathrm{RCD}(K,\infty)$ spaces and to prove that the resulting Ricci curvature tensor, now measure-valued, is bounded from below by K. Although giving comparable results, we remark that the definitions used in [23] are different from those in [42]: it is indeed unclear how to give a meaning to formula (5.5) in the non-smooth setting, so that in [23] the definition (5.3) has been adopted.

Gigli worked solely in the $\mathrm{RCD}(K,\infty)$ setting. The contribution of the current work is to adapt Gigli's tool and Sturm's computations to give a complete description of the N-Ricci curvature tensor on $\mathrm{RCD}^*(K,N)$ spaces for $N<\infty$.

Our main result is the fact that the N-Ricci curvature is bounded from below by K on a $\mathrm{RCD}(K',\infty)$ space if and only if the space is $\mathrm{RCD}^*(K,N)$.

5.2 Preliminaries

Let $M=(X,\mathrm{d},\mathfrak{m})$ be a $\mathrm{RCD}(K,\infty)$ metric measure space for some $K\in\mathbb{R}$ (or for simply, a RCD space). We denote the space of finite Borel measures on X by $\mathrm{Meas}(M)$, and equip it with the total variation norm $\|\cdot\|_{\mathrm{TV}}$.

The Sobolev space $W^{1,2}(M)$ is defined as in [8], and the weak gradient of a function $f\in W^{1,2}(M)$ is denoted by $|\mathrm{D}f|$. It is part of the definition of $\mathrm{RCD}(K,\infty)$ space that $W^{1,2}(M)$ is a Hilbert space, in which case $(X,\mathrm{d},\mathfrak{m})$ is called infinitesimally Hilbertian space. In order to introduce the concepts of 'tangent/cotangent vector field' in non-smooth setting, we will use the vocabulary of L^∞-module.

Definition 5.1 (L^2-normed L^∞-module)**.** Let $M=(X,\mathrm{d},\mathfrak{m})$ be a metric measure space. A L^2-normed $L^\infty(M)$ module is a Banach space $(\mathbf{B},\|\cdot\|_{\mathbf{B}})$ equipped with a bilinear map

$$\begin{aligned} L^\infty(M)\times\mathbf{B} &\mapsto \mathbf{B},\\ (f,v) &\mapsto f\cdot v \end{aligned}$$

such that

$$\begin{aligned} (fg)\cdot v &= f\cdot(g\cdot v),\\ \mathbf{1}\cdot v &= v \end{aligned}$$

for every $v\in\mathbf{B}$ and $f,g\in L^\infty(M)$, where $\mathbf{1}\in L^\infty(M)$ is the function identically equals to 1 on X, and a 'pointwise norm' $|\cdot|:\mathbf{B}\mapsto L^2(M)$ which maps $v\in\mathbf{B}$ to a non-negative function in $L^2(M)$ such that

$$\begin{aligned} \|v\|_{\mathbf{B}} &= \||v|\|_{L^2}\\ |f\cdot v| &= |f||v|,\quad \mathfrak{m}-\text{a.e.} \end{aligned}$$

for every $f\in L^\infty(M)$ and $v\in\mathbf{B}$.

Now we define the tangent and cotangent modules of M which are particular examples of L^2-normed module. We define the 'Pre-Cotangent Module' $\mathcal{PCM}$ as the set consisting the elements with the from $\{(A_i, f_i)\}_{i\in\mathbb{N}}$, where $\{A_i\}_{i\in\mathbb{N}}$ is a Borel partition of X, and $\{f_i\}_i$ are Sobolev functions such that $\sum_i \int_{A_i} |\mathrm{D}f_i|^2 < \infty$.

We define an equivalence relation on $\mathcal{PCM}$ via

$$\{(A_i, f_i)\}_{i\in\mathbb{N}} \sim \{(B_j, g_j)\}_{j\in\mathbb{N}} \quad \text{if} \quad |\mathrm{D}(g_j - f_i)| = 0, \ \ \mathfrak{m}-\text{a.e. on } A_i \cap B_j.$$

We denote the equivalence class of $\{(A_i, f_i)\}_{i\in\mathbb{N}}$ by $[(A_i, f_i)]$. In particular, we call $[(X, f)]$ the differential of a Sobolev function f and denote it by $\mathrm{d}f$.

Then we define the following operations:

1) $[(A_i, f_i)] + [(B_i, g_i)] := [(A_i \cap B_j, f_i + g_j)]$,

2) Multiplication by scalars: $\lambda[(A_i, f_i)] := [(A_i, \lambda f_i)]$,

3) Multiplication by simple functions: $(\sum_j \lambda_j \chi_{B_j})[(A_i, f_i)] := [(A_i \cap B_j, \lambda_j f_i)]$,

4) Pointwise norm: $|[(A_i, f_i)]| := \sum_i \chi_{A_i} |\mathrm{D}f_i|$,

where χ_A denote the characteristic function on the set A.

It can be seen that all the operations above are continuous on $\mathcal{PCM}/\sim$ with respect to the norm $\|[(A_i, f_i)]\| := \sqrt{\int |[(A_i, f_i)]|^2\,\mathfrak{m}}$ and the $L^\infty(M)$-norm on the space of simple functions. Therefore we can extend them to the completion of $(\mathcal{PCM}/\sim, \|\cdot\|)$ and we denote this completion by $L^2(T^*M)$. As a consequence of our definition, we can see that $L^2(T^*M)$ is the $\|\cdot\|$ closure of $\{\sum_{i\in I} a_i \mathrm{d}f_i : |I| < \infty, a_i \in L^\infty(M), f_i \in W^{1,2}\}$ (see Proposition 2.2.5 in [23] for a proof). It can also be seen from the definition and the infinitesimal Hilbertianity assumption on M that $L^2(T^*M)$ is a Hilbert space equipped with the inner product induced by $\|\cdot\|$. Moreover, $(L^2(T^*M), \|\cdot\|, |\cdot|)$ is a L^2-normed module according to the Definition 5.1, which we shall call cotangent module of M.

We then define the tangent module $L^2(TM)$ as $\mathrm{Hom}_{L^\infty(M)}(L^2(T^*M), L^1(M))$, i.e. $T \in L^2(T^*M)$ if it is a continuous linear map from $L^2(T^*M)$ to $L^!(M)$ viewed as Banach spaces satisfying the homogeneity:

$$T(fv) = fT(v), \quad \forall v \in L^2(T^*M), \quad f \in L^\infty(M).$$

It can be seen that $L^2(TM)$ has a natural L^2-normed $L^\infty(M)$-module structure and is isometric to $L^2(T^*M)$ both as a module and as a Hilbert space. We denote the corresponding element of $\mathrm{d}f$ in $L^2(TM)$ by ∇f and call it the gradient of f (see also the Riesz theorem for Hilbert modules in Chapter 1 of [23]). The natural pointwise norm on $L^2(TM)$ (we also denote it by $|\cdot|$) satisfies $|\nabla f| = |\mathrm{d}f| = |\mathrm{D}f|$. We can also prove that $\{\sum_{i\in I} a_i \nabla f_i : |I| < \infty, a_i \in L^\infty(M), f_i \in W^{1,2}\}$ is dense in $L^2(TM)$.

In other words, since we have a pointwise inner product $\langle\cdot,\cdot\rangle : [L^2(T^*M)]^2 \mapsto L^1(M)$ satisfying

$$\langle \mathrm{d}f, \mathrm{d}g\rangle := \frac{1}{4}(|\mathrm{D}(f+g)|^2 - |\mathrm{D}(f-g)|^2)$$

for $f, g \in W^{1,2}(M)$. We can define the gradient ∇g as the element in $L^2(TM)$ such that $\nabla g(\mathrm{d}f) = \langle \mathrm{d}f, \mathrm{d}g\rangle, \mathfrak{m}$-a.e. for every $f \in W^{1,2}(M)$. Therefore, $L^2(TM)$ inherits a pointwise inner product from $L^2(T^*M)$ and we still use $\langle\cdot,\cdot\rangle$ to denote it.

Then we can define the Laplacian by duality (integration by part) as on a Riemannian manifold.

Definition 5.2 (Measure valued Laplacian, [23, 24])**.** The space $\mathrm{D}(\mathbf{\Delta}) \subset W^{1,2}(M)$ is the space of $f \in W^{1,2}(M)$ such that there is a measure μ satisfying

$$\int h\,\mu = -\int \langle \nabla h, \nabla f\rangle\,\mathfrak{m}, \forall h : M \mapsto \mathbb{R}, \quad \text{Lipschitz with bounded support.}$$

In this case the measure μ is unique and we shall denote it by $\mathbf{\Delta} f$. If $\mathbf{\Delta} f \ll m$, we denote its density by Δf.

We define $\mathrm{TestF}(M) \subset W^{1,2}(M)$, the set of test functions by

$$\mathrm{TestF}(M) := \{f \in \mathrm{D}(\mathbf{\Delta}) \cap L^\infty : |\mathrm{D}f| \in L^\infty \quad \text{and} \quad \Delta f \in W^{1,2}(M)\}.$$

It is known from [9] that $\mathrm{TestF}(M)$ is dense in $W^{1,2}(M)$ when M is RCD.

It is proved in [39] that $\langle \nabla f, \nabla g\rangle \in \mathrm{D}(\mathbf{\Delta}) \subset W^{1,2}(M)$ for any $f, g \in \mathrm{TestF}(M)$. Therefore we can define the Hessian and Γ_2 operator as follows.

Let $f \in \mathrm{TestF}(M)$. We define the Hessian $\mathrm{H}_f : \{\nabla g : g \in \mathrm{TestF}(M)\}^2 \mapsto L^0(M)$ by

$$2\mathrm{H}_f(\nabla g, \nabla h) = \langle \nabla g, \nabla\langle \nabla f, \nabla h\rangle\rangle + \langle \nabla h, \nabla\langle \nabla f, \nabla g\rangle\rangle - \langle \nabla f, \nabla\langle \nabla g, \nabla h\rangle\rangle$$

for any $g, h \in \mathrm{TestF}(M)$. It can be seen that H_f can be extended to a symmetric $L^\infty(M)$-bilinear map on $L^2(TM)$ and continuous with values in $L^0(M)$.

Let $f, g \in \mathrm{TestF}(M)$. We define the measure $\mathbf{\Gamma}_2(f, g)$ as

$$\mathbf{\Gamma}_2(f,g) = \frac{1}{2}\mathbf{\Delta}\langle \nabla f, \nabla g\rangle - \frac{1}{2}\big(\langle \nabla f, \nabla \Delta g\rangle + \langle \nabla g, \nabla \Delta f\rangle\big)\,\mathfrak{m},$$

and we put $\mathbf{\Gamma}_2(f) := \mathbf{\Gamma}_2(f,f)$.

Then we recall some results on the non-smooth Bakry-Émery theory.

Proposition 5.3 (Bakry-Émery condition, [7], [21])**.** *Let* $M = (X, \mathrm{d}, \mathfrak{m})$ *be a* RCD *space. Then it is a* RCD$^*(K,N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1,\infty]$ *if and only if*

$$\mathbf{\Gamma}_2(f) \geq \left(K|\mathrm{D}f|^2 + \frac{1}{N}(\Delta f)^2\right)\mathfrak{m}$$

for any $f \in \mathrm{TestF}(M)$.

Lemma 5.4 ([39])**.** *Let* $M = (X, \mathrm{d}, \mathfrak{m})$ *be a* RCD$^*(K,N)$ *space,* $n \in \mathbb{N}$*,* $f_1, ..., f_n \in \mathrm{TestF}(M)$ *and* $\Phi \in C^\infty(\mathbb{R}^n)$ *be with* $\Phi(0) = 0$*. Put* $\mathbf{f} = (f_1, ..., f_n)$*, then* $\Phi(\mathbf{f}) \in \mathrm{TestF}(M)$*. In particular,* $\mathbf{\Gamma}_2(\Phi(\mathbf{f})) \geq \left(K|\mathrm{D}\Phi(\mathbf{f})|^2 + \frac{(\Delta\Phi(\mathbf{f}))^2}{N}\right)\mathfrak{m}$.

Lemma 5.5 (Chain rules, [14], [39])**.** *Let* $f_1, ..., f_n \in \mathrm{TestF}(M)$ *and* $\Phi \in C^\infty(\mathbb{R}^n)$ *be with* $\Phi(0) = 0$*. Put* $\mathbf{f} = (f_1, ..., f_n)$*, then*

$$|\mathrm{D}\Phi(\mathbf{f})|^2\,\mathfrak{m} = \sum_{i,j=1}^n \Phi_i\Phi_j(\mathbf{f})\langle\nabla f_i, \nabla f_j\rangle\,\mathfrak{m},$$

$$\begin{aligned}\mathbf{\Gamma}_2(\Phi(\mathbf{f})) &= \sum_{i,j}\Phi_i\Phi_j(\mathbf{f})\mathbf{\Gamma}_2(f_i,f_j)\\ &+ 2\sum_{i,j,k}\Phi_i\Phi_{j,k}(\mathbf{f})\mathrm{H}_{f_i}(\nabla f_j, \nabla f_k)\,\mathfrak{m}\\ &+ \sum_{i,j,k,l}\Phi_{i,j}\Phi_{k,l}(\mathbf{f})\langle\nabla f_i,\nabla f_k\rangle\langle\nabla f_j,\nabla f_l\rangle\,\mathfrak{m},\end{aligned}$$

and

$$\mathbf{\Delta}\Phi(\mathbf{f}) = \sum_{i=1}^n \Phi_i(\mathbf{f})\mathbf{\Delta}f_i + \sum_{i,j=1}^n \Phi_{i,j}(\mathbf{f})\langle\nabla f_i,\nabla f_j\rangle\,\mathfrak{m}.$$

At the end of this section, we discuss the dimension of M which is understood as the dimension of $L^2(TM)$ as a L^∞-module. Let A be a Borel set. We denote the subset of $L^2(TM)$ consisting those v such that $\chi_{A^c}v = 0$ by $L^2(TM)_{|_A}$.

Definition 5.6 (Local independence)**.** Let A be a Borel set with positive measure. We say that $\{v_i\}_1^n \subset L^2(TM)$ is independent on A if

$$\sum_i f_i v_i = 0, \quad \mathfrak{m} - \text{a.e. on } A$$

holds if and only if $f_i = 0$ $\mathfrak{m}$-a.e. on A for each i.

Definition 5.7 (Local span and generators)**.** Let A be a Borel set in X and $V := \{v_i\}_{i\in I} \subset L^2(TM)$. The span of V on A, denoted by $\mathrm{Span}_A(V)$, is the subset of

$L^2(TM)_{|A}$ with the following property: there exist a Borel decomposition $\{A_n\}_{n\in\mathbb{N}}$ of A and families of vectors $\{v_{i,n}\}_{i=1}^{m_n} \subset L^2(TM)$ and functions $\{f_{i,n}\}_{i=1}^{m_n} \subset L^\infty(M)$, $n = 1, 2, ...$, such that

$$\chi_{A_n} v = \sum_{i=1}^{m_n} f_{i,n} v_{i,n}$$

for each n. We call the closure of $\mathrm{Span}_A(V)$ the space generated by V on A.

We say that $L^2(TM)$ is finitely generated if there is a finite family $v_1, ..., v_n$ spanning $L^2(TM)$ on X, and locally finitely generated if there is a partition $\{E_i\}$ of X such that $L^2(TM)_{|E_i}$ is finitely generated for every $i \in \mathbb{N}$.

Definition 5.8 (Local basis and dimension)**.** We say that a finite set $v_1, ..., v_n$ is a basis on Borel set A if it is independent on A and $\mathrm{Span}_A\{v_1, ..., v_n\} = L^2(TM)_{|A}$. If $L^2(TM)$ has a basis of cardinality n on A, we say that it has dimension n on A, or that its local dimension on A is n. If $L^2(TM)$ does not admit any local basis of finite cardinality on any subset of A with positive measure, we say that $L^2(TM)$ has infinite dimension on A.

It can be proved (see Proposition 1.4.4 in [23] for example) that the definition of basis and dimension are well posed. As a consequence of this definition, we can prove the existence of a unique decomposition $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ of X such that for each E_n with positive measure, $n \in \mathbb{N}\cup\{\infty\}$, $L^2(TM)$ has dimension n on E_n. Furthermore, thanks to the infinitesimal Hilbertianity we have the following proposition.

Proposition 5.9 (Theorem 1.4.11, [23])**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}(K,\infty)$ *metric measure space. Then there exists a unique decomposition* $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ *of* X *such that*

- *For any* $n \in \mathbb{N}$ *and any* $B \subset E_n$ *with finite positive measure,* $L^2(TM)$ *has a unit orthogonal basis* $\{e_{i,n}\}_{i=1}^n$ *on* B*,*
- *For every subset* B *of* E_∞ *with finite positive measure, there exists a unit orthogonal set* $\{e_{i,B}\}_{i\in\mathbb{N}\cup\{\infty\}} \subset L^2(TM)_{|B}$ *which generates* $L^2(TM)_{|B}$*,*

where unit orthogonal of a countable set $\{v_i\}_i \subset L^2(TM)$ *on* B *means* $\langle v_i, v_j\rangle = \delta_{ij}$ $\mathfrak{m}$*-a.e. on* B*.*

Definition 5.10 (Global Dimension)**.** We say that the dimension of $L^2(TM)$ is k if $k = \sup\{n : \mathfrak{m}(E_n) > 0\}$ where $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ is the decomposition given in Proposition 5.9. We define the dimension of M as the dimension of $L^2(TM)$ and denote it by $\dim M$.

5.3 Improved Bochner inequality

In this part, we will study the dimension of $\mathrm{RCD}^*(K,N)$ metric measure spaces and prove an improved Bochner inequality.

First of all, we have a lemma.

Lemma 5.11 (Lemma 3.3.6, [23]). *Let* $\mu_i = \rho_i\,\mathfrak{m} + \mu_i^s, i = 1,2,3$ *be measures with* $\mu_i^s \perp \mathfrak{m}$. *We assume that*

$$\lambda^2\mu_1 + 2\lambda\mu_2 + \mu_3 \geq 0, \qquad \forall\lambda \in \mathbb{R}.$$

Then we have

$$\mu_1^s \geq 0, \quad \mu_3^s \geq 0$$

and

$$|\rho_2|^2 \leq \rho_1\rho_3, \quad \mathfrak{m} - a.e..$$

Now we prove that N is an upper bound of the dimensions of $\mathrm{RCD}^*(K,N)$ spaces.

Proposition 5.12. *Let* $M = (X, \mathrm{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}^*(K,N)$ *metric measure space. Then* $\dim M \leq N$. *Furthermore, if the local dimension on a Borel set* E *is* N, *we have* $\mathrm{tr}\mathrm{H}_f(x) = \Delta f(x)$ $\mathfrak{m}$*-a.e.* $x \in E$ *for every* $f \in \mathrm{TestF}$.

Proof. Let $\{E_m\}_{m\in\mathbb{N}\cup\{\infty\}}$ be the partition of X given by Proposition 5.9. To prove $\dim M \leq N$, it is sufficient to prove that for any E_m with positive measure, we have $m \leq N$.

Then, let $m \in \mathbb{N}\cup\{\infty\}$ be such that $\mathfrak{m}(E_m) > 0$, and $n \leq m$ a finite number. We define the function $\Phi(x,y,z_1,...,z_n) := \lambda(xy+x) - by + \sum_i^n(z_i - c_i)^2 - \sum_i^n c_i^2$ where $\lambda, b, c_i \in \mathbb{R}$. Then we have

$$\begin{aligned} \Phi_{x,i} &= 0, \qquad \Phi_{y,i} = 0, \qquad \Phi_{i,j} = 2\delta_{ij}, \qquad \Phi_{x,y} = \lambda \\ \Phi_x &= \lambda y + \lambda, \qquad \Phi_y = \lambda x - b, \qquad \Phi_i = 2(z_i - c_i). \end{aligned}$$

From Lemma 5.4 we know

$$\mathbf{\Gamma}_2(\Phi(\mathbf{f})) \geq \left(K|\mathrm{D}\Phi(\mathbf{f})|^2 + \frac{(\Delta\Phi(\mathbf{f}))^2}{N}\right)\mathfrak{m}$$

for any $\mathbf{f} = (f, g, h_1, ..., h_n)$ where $f, g, h_1, ..., h_n \in \mathrm{TestF}$.

Combining the chain rules (see Lemma 5.5), the inequality above becomes:

$$\mathbf{A}(\lambda,b,\mathbf{c}) \geq \Big(KB(\lambda,b,\mathbf{c})+\frac{1}{N}C^2(\lambda,b,\mathbf{c})\Big)\,\mathfrak{m}, \tag{5.6}$$

where

$$\begin{aligned}
\mathbf{A}(\lambda,b,\mathbf{c}) &= (\lambda f-b)^2\mathbf{\Gamma}_2(g)+(\lambda g+\lambda)^2\mathbf{\Gamma}_2(f)+\sum_{i,j}4(h_i-c_i)(h_j-c_j)\mathbf{\Gamma}_2(h_i,h_j)\\
&+ 2\lambda(g+1)(\lambda f-b)\mathbf{\Gamma}_2(f,g)+\sum_i 4(\lambda g+\lambda)(h_i-c_i)\mathbf{\Gamma}_2(f,h_i)\\
&+ \sum_i 4(\lambda f-b)(h_i-c_i)\mathbf{\Gamma}_2(g,h_i)+8\lambda\sum_i(h_i-c_i)\mathrm{H}_{h_i}(\nabla f,\nabla g)\,\mathfrak{m}\\
&+ 4\sum_i(\lambda g+\lambda)\mathrm{H}_f(\nabla h_i,\nabla h_i)\,\mathfrak{m}+4\sum_i(\lambda f-b)\mathrm{H}_g(\nabla h_i,\nabla h_i)\,\mathfrak{m}\\
&+ 4\lambda(\lambda f-b)\mathrm{H}_g(\nabla f,\nabla g)\,\mathfrak{m}+4\lambda(\lambda g+\lambda)\mathrm{H}_f(\nabla f,\nabla g)\,\mathfrak{m}\\
&+ 8\sum_{i,j}(h_i-c_i)\mathrm{H}_{h_i}(\nabla h_j,\nabla h_j)\,\mathfrak{m}+2\lambda^2|\mathrm{D}f|^2|\mathrm{D}g|^2\,\mathfrak{m}+2\lambda^2|\langle\nabla f,\nabla g\rangle|^2\,\mathfrak{m}\\
&+ 4\sum_{i,j}|\langle\nabla h_i,\nabla h_j\rangle|^2\,\mathfrak{m}+8\lambda\sum_i\langle\nabla f,\nabla h_i\rangle\langle\nabla g,\nabla h_i\rangle\,\mathfrak{m}\\
B(\lambda,b,\mathbf{c}) &= (\lambda f-b)^2|\mathrm{D}g|^2+(\lambda g+\lambda)^2|\mathrm{D}f|^2\\
&+ 2(\lambda g+\lambda)(\lambda f-b)\langle\nabla f,\nabla g\rangle+4\sum_i(\lambda g+\lambda)(h_i-c_i)\langle\nabla f,\nabla h_i\rangle\\
&+ 4\sum_i(\lambda f-b)(h_i-c_i)\langle\nabla g,\nabla h_i\rangle+4\sum_{i,j}(h_i-c_i)(h_j-c_j)\langle\nabla h_i,\nabla h_j\rangle\\
C(\lambda,b,\mathbf{c}) &= (\lambda g+\lambda)\Delta f+(\lambda f-b)\Delta g+2\sum_i(h_i-c_i)\Delta h_i\\
&+ 2\lambda\langle\nabla f,\nabla g\rangle+2\sum_i|\mathrm{D}h_i|^2.
\end{aligned}$$

Let B be an arbitrary Borel set. From the inequality (5.6) we know

$$\chi_B\mathbf{A}(\lambda,b,\mathbf{c}) \geq \Big(K\chi_B B(\lambda,b,\mathbf{c})+\frac{1}{N}\chi_B C^2(\lambda,b,\mathbf{c})\Big)\,\mathfrak{m}.$$

Combining this observation and the linearity of $\mathbf{A},B,C$ with respect to b, we can replace the constant b in (5.6) by an arbitrary simple function. Pick a sequence of simple functions $\{b_n\}_n$ such that $b_n\to\lambda f$ in $L^\infty(M)$. Since $\mathbf{\Gamma}_2(f,g)$ and $\mathrm{H}_f(\nabla g,\nabla h)\,\mathfrak{m}$ have finite total variation for any $f,g,h\in\mathrm{TestF}$, we can see that $\mathbf{A}(\lambda,b_n,\mathbf{c})$, $B(\lambda,b_n,\mathbf{c})\mathfrak{m}$, $C^2(\lambda,b_n,\mathbf{c})\mathfrak{m}$ converge to $\mathbf{A}(\lambda,\lambda f,\mathbf{c})$, $B(\lambda,\lambda f,\mathbf{c})\mathfrak{m}$, $C^2(\lambda,\lambda f,\mathbf{c})\mathfrak{m}$ respectively with respect to the total variation norm $\|\cdot\|_{\mathrm{TV}}$. Therefore, we can replace b in (5.6) by λf. For the same reason, we can replace c_i by h_i. Then we obtain the following inequality.

$$\mathbf{A}'(\lambda) \geq \Big(KB'(\lambda)+\frac{1}{N}(C')^2(\lambda)\Big)\,\mathfrak{m}, \tag{5.7}$$

where

$$
\begin{aligned}
\mathbf{A}'(\lambda) &= 4\sum_i (\lambda g+\lambda)\mathrm{H}_f(\nabla h_i, \nabla h_i)\,\mathfrak{m} + 2\lambda^2|\mathrm{D}f|^2|\mathrm{D}g|^2\,\mathfrak{m} + 2\lambda^2|\langle \nabla f, \nabla g\rangle|^2\,\mathfrak{m} \\
&+ 4\sum_{i,j}|\langle \nabla h_i, \nabla h_j\rangle|^2\,\mathfrak{m} + 8\lambda\sum_i \langle \nabla f, \nabla h_i\rangle\langle \nabla g, \nabla h_i\rangle\,\mathfrak{m} + (\lambda g+\lambda)^2\mathbf{\Gamma}_2(f) \\
&+ 4\lambda(\lambda g+\lambda)\mathrm{H}_f(\nabla f, \nabla g) \\
B'(\lambda) &= (\lambda g+\lambda)^2|\mathrm{D}f|^2 \\
C'(\lambda) &= (\lambda g+\lambda)\Delta f + 2\lambda\langle \nabla f, \nabla g\rangle + 2\sum_i |\mathrm{D}h_i|^2.
\end{aligned}
$$

It can be seen that $\mathrm{H}_f(\nabla f, \nabla g) = \frac{1}{2}\langle \nabla |\mathrm{D}f|^2, \nabla g\rangle$. Therefore, all the terms in $\mathbf{A}', B'\,\mathfrak{m}$ and $C'\,\mathfrak{m}$ vary continuously w.r.t. $\|\cdot\|_{\mathrm{TV}}$ as g varies in $W^{1,2}(M)$. Hence the inequality (5.7) holds for any Lipschitz function g with bounded support. In particular, we can pick g identically 1 on some bounded set $\Omega \subset X$, so that we have $|\mathrm{D}g| = 0$ and $\mathrm{H}_f(\nabla f, \nabla g) = 0$ $\mathfrak{m}$-a.e. on Ω. By the arbitrariness of Ω we can replace g by $\mathbf{1}$ which is the function identically equals to 1 on X. Then the inequality (5.7) becomes:

$$
\begin{aligned}
&\lambda^2\mathbf{\Gamma}_2(f) + \Big(2\lambda\sum_i \mathrm{H}_f(\nabla h_i, \nabla h_i) + \sum_{i,j}|\langle \nabla h_i, \nabla h_j\rangle|^2 - K\lambda^2|\mathrm{D}f|^2\Big)\mathfrak{m} \\
-&\Big(\lambda^2\frac{(\Delta f)^2}{N} + 2\lambda\frac{\Delta f}{N}|\mathrm{D}h_i|^2 + \frac{(\sum_i |\mathrm{D}h_i|^2)^2}{N}\Big)\mathfrak{m} \geq 0.
\end{aligned}
$$

Let $\gamma_2(f)\,\mathfrak{m}$ be the absolutely continuous part of $\mathbf{\Gamma}_2(f)$. By Lemma 5.11 we have the inequality

$$
\begin{aligned}
&\left|\sum_i \Big(\mathrm{H}_f(\nabla h_i, \nabla h_i) - \frac{\Delta f}{N}|\mathrm{D}h_i|^2\Big)\right|^2 \\
\leq\ &\Big(\gamma_2(f) - K|\mathrm{D}f|^2 - \frac{(\Delta f)^2}{N}\Big)\Big(\sum_{i,j}|\langle \nabla h_i, \nabla h_j\rangle|^2 - \frac{(\sum_i |\mathrm{D}h_i|^2)^2}{N}\Big).
\end{aligned}
$$

In particular, since $\gamma_2(f) - K|\mathrm{D}f|^2 - \frac{(\Delta f)^2}{N} \geq 0$ (by Proposition 5.3), we have

$$
\sum_{i,j}|\langle \nabla h_i, \nabla h_j\rangle|^2 \geq \frac{(\sum_i |\mathrm{D}h_i|^2)^2}{N}, \qquad \mathfrak{m}-\text{a.e.}.
$$

This inequality remains true if we replace ∇h_i by $v := \sum_k \chi_{A_k}\nabla f_k$ where f_k are test functions and A_k are disjoint Borel sets. Therefore by density we can replace $\{\nabla h_i\}_1^n$ by any $\{e_{i,m}\}_{i=1}^n$ which is a unit orthogonal subset of $L^2(TM)|_{E_m}$, whose existence is guaranteed by Proposition 5.9 and the choice of m, n, E_m at the beginning of the proof. Then we obtain

$$n = \sum_{i,j} |\langle e_{i,m}, e_{j,m}\rangle|^2 \geq \frac{(\sum_i |e_{i,m}|^2)^2}{N} = \frac{n^2}{N}, \qquad \mathfrak{m} - \text{a.e. on} \quad E_m,$$

which implies $n \leq N$ on E_m. Since the finite integer $n \leq m$ was chosen arbitrarily, we deduce $m \leq N$. Furthermore, if E_N has positive measure, we obtain

$$\left| \sum_{i=1}^N \mathrm{H}_f(e_{i,N}, e_{i,N}) - \sum_{i=1}^N \frac{\Delta f}{N} |e_{i,N}|^2 \right| = 0, \qquad \mathfrak{m} - \text{a.e. on } E_N,$$

where $\{e_{i,N}\}_{i=1}^N$ is a unit orthogonal basis on E_N. This is the same as to say that $\mathrm{trH}_f = \Delta f$, $\mathfrak{m}$-a.e. on E.

□

According to this proposition, on RCD$^*(K,N)$ spaces, we can see that the pointwise Hilbert-Schmidt norm $\|T\|_{\mathrm{HS}}$ of a L^∞-bilinear map $T : [L^2(TM)]^2 \mapsto L^0(M)$ is always well defined by the following procedure. We denote $\dim_{\mathrm{loc}} : M \mapsto \mathbb{N}$ as the local dimension which is defined as $\dim_{\mathrm{loc}}(x) = n$ on E_n, where $\{E_n\}_{n \in \mathbb{N} \cup \{\infty\}}$ is the partition of X in Proposition 5.9. Let $T_1, T_2 : [L^2(TM)]^2 \mapsto L^0(M)$ be symmetric bilinear maps, we define $\langle T_1, T_2\rangle_{\mathrm{HS}}$ as a function such that $\langle T_1, T_2\rangle_{\mathrm{HS}} := \sum_{i,j} T_1(e_{i,n}, e_{j,n}) T_2(e_{i,n}, e_{j,n})$, $\mathfrak{m}$-a.e. on E_n, where $\{E_n\}_{n \leq N}$ is the partition of X in Proposition 5.9 and $\{e_{i,n}\}_i, n = 1, ..., \lfloor N \rfloor$ are the corresponding unit orthogonal basis. Clearly, this definition is well posed. In particular, we define the Hilbert-Schmidt norm of T_1 by $\sqrt{\langle T_1, T_1\rangle_{\mathrm{HS}}}$ and denote it by $\|T_1\|_{\mathrm{HS}}$, and the trace of T_1 can be written in the way that $\mathrm{tr} T_1 = \langle T_1, \mathrm{Id}_{\dim_{\mathrm{loc}}}\rangle_{\mathrm{HS}}$ where $\mathrm{Id}_{\dim_{\mathrm{loc}}}$ is the unique map satisfying $\mathrm{Id}_{\dim_{\mathrm{loc}}}(e_{i,\dim_{\mathrm{loc}}}, e_{j,\dim_{\mathrm{loc}}}) = \delta_{ij}$, $\mathfrak{m}$-a.e. on $E_{\dim_{\mathrm{loc}}}$.

In the following theorem, we prove an improved Bochner inequality.

Theorem 5.13. *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a* RCD$^*(K,N)$ *metric measure space. Then*

$$\mathbf{\Gamma}_2(f) \quad \geq \quad \Big(K|\mathrm{D}f|^2 + \|\mathrm{H}_f\|_{\mathrm{HS}}^2 + \frac{1}{N - \dim_{\mathrm{loc}}} (\mathrm{trH}_f - \Delta f)^2 \Big)\, \mathfrak{m}$$

holds for any $f \in$ TestF, *where $\frac{1}{N - \dim_{\mathrm{loc}}}(\mathrm{trH}_f - \Delta f)^2)$ is taken 0 by definition on the set $\{x : \dim_{\mathrm{loc}}(x) = N\}$.*

Proof. We define the function Φ as

$$\Phi(x, y_1, ..., y_N) := x - \frac{1}{2} \sum_{i,j}^N c_{i,j}(y_i - c_i)(y_j - c_j) - c \sum_{i=1}^N (y_i - c_i)^2 + C_0$$

where $c, c_i, c_{i,j} = c_{j,i}$ are constants, $C_0 = \frac{1}{2}\sum_{i,j}^N c_{i,j}c_ic_j + c\sum_{i=1}^N c_i^2$. Then we have

$$\begin{aligned}\Phi_{x,i} = \Phi_{x,x} &= 0, \qquad \Phi_{i,j} = -c_{i,j} - 2c\delta_{ij}\\ \Phi_x &= 1, \qquad \Phi_i = -\sum_j c_{i,j}(y_j - c_j) - 2c(y_i - c_i).\end{aligned}$$

Let $f, h_1, ..., , h_N$ be test functions. Using the chain rules we have

$$\begin{aligned}&|\mathrm{D}\Phi(f, h_1, ..., h_N)|^2 = |\mathrm{D}f|^2 + \sum_i (h_i - c_i)I_i,\\ &\mathbf{\Gamma}_2(\Phi(f, h_1, ..., h_N)) = \mathbf{\Gamma}_2(f) - 2\sum_{i,j}(c_{i,j} + 2c\delta_{ij})\mathrm{H_f}(\nabla \mathrm{h_i}, \nabla \mathrm{h_j})\,\mathfrak{m}\\ &-\sum_{i,j,k,l}(c_{i,j} + 2c\delta_{ij})(c_{k,l} + 2c\delta_{kl})\langle \nabla h_i, \nabla h_k\rangle\langle \nabla h_l, \nabla h_j\rangle\,\mathfrak{m} + \sum_i (h_i - c_i)\mathbf{J}_i,\\ &\Delta\Phi(f, h_1, ..., h_N) = \Delta f - \sum_{i,j}(c_{i,j} + 2c\delta_{ij})\langle \nabla h_i, \nabla h_j\rangle + \sum_i (h_i - c_i)K_i\end{aligned}$$

where $\{I_i, K_i\}_i$ are some $L^1(M)$-integrable terms and $\{\mathbf{J}_i\}_i$ are measures with finite mass.

Then we apply Lemma 5.4 to the function $\Phi(f, h_1, ..., h_N)$ to obtain the inequality

$$\begin{aligned}&\mathbf{\Gamma}_2(f) - 2\sum_{i,j}(c_{i,j} + 2c\delta_{ij})\mathrm{H}_f(\nabla h_i, \nabla h_j)\,\mathfrak{m}\\ -\ &\sum_{i,j,k,l}(c_{i,j} + 2c\delta_{ij})(c_{k,l} + 2c\delta_{kl})\langle \nabla h_i, \nabla h_k\rangle\langle \nabla h_l, \nabla h_j\rangle\,\mathfrak{m} + \sum_i (h_i - c_i)J_i\,\mathfrak{m}\\ \geq\ &K\Big(|\mathrm{D}f|^2 + \sum_i (h_i - c_i)I_i\Big)\,\mathfrak{m} +\\ &\frac{1}{N}\Big(\Delta(f) - \sum_{i,j}(c_{i,j} + 2c\delta_{ij})\langle \nabla h_i, \nabla h_j\rangle + \sum_i (h_i - c_i)K_i\Big)^2\,\mathfrak{m}.\end{aligned}$$

Using the same argument as in the proof of last Proposition, we can replace the constants c_i by any simple function. Furthermore, by an approximation argument we can replace the constant $c, c_i, c_{i,j}$ by arbitrary L^2 functions. Then pick $c_i = h_i$, the inequality becomes

$$\begin{aligned} &\mathbf{\Gamma}_2(f) - 2\sum_{i,j}(c_{i,j}+2c\delta_{ij})\mathrm{H}_f(\nabla h_i,\nabla h_j)\,\mathfrak{m} \\ - &\sum_{i,j,k,l}(c_{i,j}+2c\delta_{ij})(c_{k,l}+2c\delta_{kl})\langle\nabla h_i,\nabla h_k\rangle\langle\nabla h_l,\nabla h_j\rangle\,\mathfrak{m} \\ \geq\ & K|\mathrm{D}f|^2\,\mathfrak{m} + \frac{1}{N}\Big(\Delta(f) - \sum_{i,j}(c_{i,j}+2c\delta_{ij})\langle\nabla h_i,\nabla h_j\rangle\Big)^2\,\mathfrak{m}. \end{aligned}$$

Now we restrict the inequality above on Borel set E_n, $n\leq N$ where $\{E_n\}_{n=1}^{\lfloor N\rfloor}$ is the partition of X in Proposition 5.9. The inequality remains true if we replace ∇h_i by $v:=\sum_k \chi_{A_k}\nabla f_k$ where f_k are test functions and A_k are disjoint Borel subsets of E_n. Therefore by density we can replace $\{\nabla h_i\}_1^n$ by any $\{e_{i,n}\}_{i=1}^n$ which is a unit orthogonal basis of $L^2(TM)|_{E_n}$. Doing this replacement on every E_n, we obtain

$$\begin{aligned} &\mathbf{\Gamma}_2(f) - \Big(2\sum_{i,j=1}^{\dim_{\mathrm{loc}}}(c_{i,j}+2c\delta_{ij})\mathrm{H}_f(e_{i,\dim_{\mathrm{loc}}},e_{j,\dim_{\mathrm{loc}}}) + \sum_{i,j=1}^{\dim_{\mathrm{loc}}}(c_{i,j}+2c\delta_{ij})(c_{i,j}+2c\delta_{ij})\Big)\,\mathfrak{m} \\ \geq\ & K|\mathrm{D}f|^2\,\mathfrak{m} + \frac{1}{N}\Big(\Delta(f) - \sum_{i=1}^{\dim_{\mathrm{loc}}}(c_{i,i}+2c)\Big)^2\,\mathfrak{m}. \end{aligned}$$

Pick

$$c = \begin{cases} \dfrac{N\mathrm{tr}\mathrm{H}_f - \dim_{\mathrm{loc}}\Delta f - (N-\dim_{\mathrm{loc}})\mathrm{tr}C}{2n(N-\dim_{\mathrm{loc}})} & \text{on } \{x : \dim_{\mathrm{loc}}(x)\neq N\}, \\ 0 & \text{on } \{x : \dim_{\mathrm{loc}}(x) = N\}. \end{cases}$$

in the inequality above, where $C=(c_{i,j})$ is a symmetric $\dim_{\mathrm{loc}}$-matrix, $c_{i,j}$ are L^2 functions. It can be seen (as in $\mathbb{R}^n$) that there is a one-to-one correspondance between such matrix and symmetric bilinear maps from $[L^2(TM)]^2$ to $L^2(M)$. For convenient, we still use C to denote the corresponding map of $(c_{i,j})$.

Then we obtain the inequality

$$\begin{aligned} &\mathbf{\Gamma}_2(f) \geq \Big(K|\mathrm{D}f|^2 - \|C\|_{\mathrm{HS}}^2 + 2\langle C,\mathrm{H}_f\rangle_{\mathrm{HS}}\Big)\,\mathfrak{m} \\ + &\Big(\frac{1}{N}(\mathrm{tr}C)^2 + \frac{1}{N}(\Delta f)^2 - \frac{2}{N}(\Delta f)(\mathrm{tr}C)\Big)\,\mathfrak{m} \\ + &\frac{\big((N\mathrm{tr}\mathrm{H}_f - \dim_{\mathrm{loc}}\Delta f)^2 + (\dim_{\mathrm{loc}}-N)^2(\mathrm{tr}C)^2 + 2(\dim_{\mathrm{loc}}-N)(\mathrm{tr}C)(N\mathrm{tr}\mathrm{H}_f - \dim_{\mathrm{loc}}\Delta f)\big)}{N\dim_{\mathrm{loc}}(N-\dim_{\mathrm{loc}})}\,\mathfrak{m} \\ = &\Big(K|\mathrm{D}f|^2 + \frac{1}{N}(\Delta f)^2 + \frac{1}{N\dim_{\mathrm{loc}}(N-\dim_{\mathrm{loc}})}(N\mathrm{tr}\mathrm{H}_f - \dim_{\mathrm{loc}}\Delta f)^2\Big)\,\mathfrak{m} \\ - &\Big(\|C-\mathrm{H}_f + \frac{\mathrm{tr}\mathrm{H}_f}{\dim_{\mathrm{loc}}}\mathrm{Id}_{\dim_{\mathrm{loc}}}\|_{\mathrm{HS}}^2 + \|\mathrm{H}_f - \frac{\mathrm{tr}\mathrm{H}_f}{\dim_{\mathrm{loc}}}\mathrm{Id}_{\dim_{\mathrm{loc}}}\|_{\mathrm{HS}}^2 + \frac{1}{\dim_{\mathrm{loc}}}(\mathrm{tr}C)^2\Big)\,\mathfrak{m} \end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$ is the inner product induced by the Hilbert-Schmidt norm and $\mathrm{Id}_{\mathrm{dim}_{\mathrm{loc}}}$ is the $\mathrm{dim}_{\mathrm{loc}}$-identity matrix. It can be seen from the Proposition 5.12 that this inequality still makes sense if we accept $\frac{0}{0}=0$.

Then we pick $C = \mathrm{H}_f - \frac{\mathrm{tr}\mathrm{H}_f}{\mathrm{dim}_{\mathrm{loc}}}\mathrm{Id}_{\mathrm{dim}_{\mathrm{loc}}}$ in this inequality and finally obtain

$$\begin{aligned}\mathbf{\Gamma}_2(f) &\geq \Big(K|\mathrm{D}f|^2 + \frac{1}{N}(\Delta f)^2 + \frac{1}{N\mathrm{dim}_{\mathrm{loc}}(N-\mathrm{dim}_{\mathrm{loc}})}(N\mathrm{tr}\mathrm{H}_f - \mathrm{dim}_{\mathrm{loc}}\Delta f)^2\Big)\,\mathfrak{m}\\ &+ \|\mathrm{H}_f - \frac{\mathrm{tr}\mathrm{H}_f}{\mathrm{dim}_{\mathrm{loc}}}\mathrm{Id}_{\mathrm{dim}_{\mathrm{loc}}}\|^2_{\mathrm{HS}}\,\mathfrak{m}\\ &= \Big(K|\mathrm{D}f|^2 + \|\mathrm{H}_f\|^2_{\mathrm{HS}} + \frac{1}{(N-\mathrm{dim}_{\mathrm{loc}})}(\mathrm{tr}\mathrm{H}_f - \Delta f)^2\Big)\,\mathfrak{m},\end{aligned}$$

which is the thesis.

□

5.4 N-Ricci tensor

In this section, we use the improved version of Bochner inequality that we obtained in the last section to give a definition of N-Ricci tensor.

We recall that the class of test vector fields $\mathrm{TestV}(M) \subset L^2(TM)$ is defined as

$$\mathrm{TestV}(M) := \{\sum_{i=1}^n g_i \nabla f_i : n \in \mathbb{N}, f_i, g_i \in \mathrm{TestF}(M), i = 1, ..., n\}.$$

It can proved that $\mathrm{TestV}(M)$ is dense in $L^2(TM)$ when M is RCD.

Let $X = \sum_i g_i \nabla f_i \in \mathrm{TestV}(M)$ be a test vector field. We define $\nabla X \in L^2(TM) \otimes L^2(TM)$ by the following formula:

$$\langle \nabla X, v_1 \otimes v_2\rangle_{\mathrm{HS}} := \sum_i \langle \nabla g_i, v_1\rangle\langle \nabla f_i, v_2\rangle + \sum_i g_i \mathrm{H}_{f_i}(v_1, v_2), \qquad \forall v_1, v_2 \in \mathrm{TestV}(M).$$

It can be seen that this definition is well posed and that the completion of $\mathrm{TestV}(M)$ with respect to the norm $\|\cdot\|_C := \sqrt{\|\cdot\|^2_{L^2(TM)} + \int \|\nabla\cdot\|^2_{\mathrm{HS}}\,\mathfrak{m}}$ can be identified with a subspace of $L^2(TM)$, which is denoted by $H^{1,2}_C(TM)$.

Let $X \in L^2(TM)$. We say that $X \in \mathrm{D}(\mathrm{div})$ if there exists a function $g \in L^2(M)$ such that

$$\int hg\,\mathfrak{m} = -\int \langle \nabla h, X\rangle\,\mathfrak{m}$$

for any $h \in W^{1,2}(M)$. We then denote such function which is clearly unique by $\mathrm{div}X$.

It is easy to see that div· is a linear operator on D(div), that TestV(M) $\subset$ D(div) and that the formula:

$$\mathrm{div}(g\nabla f) = \langle \nabla g, \nabla f\rangle + g\Delta f, \qquad f,g \in \mathrm{TestF}(M)$$

holds.

It is unknown whether there is any inclusion relation between D(div) and $H^{1,2}_C(TM)$. However, in [23] it has been introduced the space $(H^{1,2}_H(TM), \|\cdot\|_{H^{1,2}_H})$ which is contained in both D(div) and $H^{1,2}_C(TM)$, and will be useful for our purposes. In the smooth setting, $H^{1,2}_H(TM)$ would be the space of vector fields corresponding to L^2 1-forms having both exterior derivative and co-differential in $L^2(M)$.

The properties of $H^{1,2}_H(TM)$ that we shall need are:

(a) TestV(M) is dense in $H^{1,2}_H(TM)$,

(b) $H^{1,2}_H(TM)$ is contained in $H^{1,2}_C(TM)$ with continuous embedding,

(c) $H^{1,2}_H(TM) \subset$ D(div) and for any $X_n \to X$ in $H^{1,2}_H(TM)$, we have $\mathrm{div}X_n \to \mathrm{div}X$ in $L^2(M)$.

Now, we can generalize the Proposition 5.12 in the following way. We denote the natural correspondences (dualities) between $L^2(TM)$ and $L^2(T^*M)$ by $(\cdot)^b$ and $(\cdot)^\sharp$ (same notation for $L^2(TM)\otimes L^2(TM)$ and $L^2(T^*M)\otimes L^2(T^*M)$). For example, $(\nabla f)^b = \mathrm{d}f$, $(\mathrm{H}_f)^\sharp = \nabla\nabla f$ for $f \in \mathrm{TestF}(M)$.

Proposition 5.14. *Let* $M = (X, \mathrm{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}^*(K,N)$ *metric measure space,* $E \subset X$ *be a Borel set. Assume that the local dimension of* M *on* E *is* N*, then* $\mathrm{tr}(\nabla X)^b = \mathrm{div}X$ $\mathfrak{m}$*-a.e.* $x \in E$ *for any* $X \in H^{1,2}_H(TM)$*.*

Proof. Thanks to the Proposition 5.12, it is sufficient to prove the equality

$$\mathrm{tr}(\nabla X)^b = \mathrm{div}X \quad \mathfrak{m}-\text{a.e. on} \quad E \tag{5.8}$$

for any $X \in H^{1,2}_H(TM)$, under the assumption that $\mathrm{tr}\mathrm{H}_f = \Delta f$ $\mathfrak{m}$-a.e. on E for any $f \in \mathrm{TestF}(M)$.

First of all, as we know $(\nabla\nabla f)^b = \mathrm{H}_f$ and $\mathrm{div}(\nabla f) = \Delta f$, the equality (5.8) holds for every X of the form ∇f for some $f \in \mathrm{TestF}(M)$.

Secondly, for any $X = \sum_i g_i \nabla f_i \in \mathrm{TestV}$, the assertion holds followings recalling the identities $\nabla(g\nabla f) = \nabla g \otimes \nabla f + g\nabla(\nabla f)$ and $\mathrm{div}(g\nabla f) = \nabla g\cdot\nabla f + g\mathrm{div}(\nabla f)$.

Finally, for any $X \in H^{1,2}_H(TM)$, we can find a sequence $\{X_i\}_i \subset \mathrm{TestV}$ such that $X_i \to X$ in $H^{1,2}_H(TM)$. Therefore $\langle (\nabla X_i)^b, \mathrm{Id}_{\dim_{\mathrm{loc}}}\rangle_{\mathrm{HS}} \to \langle (\nabla X)^b, \mathrm{Id}_{\dim_{\mathrm{loc}}}\rangle_{\mathrm{HS}}$ in L^2 because $\|\cdot\|_{H^{1,2}_H}$ convergence is stronger than the $\|\cdot\|_{H^{1,2}_C}$ convergence. Then $\mathrm{tr}(\nabla X_i)^b = \langle (\nabla X_i)^b, \mathrm{Id}_{\dim_{\mathrm{loc}}}\rangle_{\mathrm{HS}} \to \langle (\nabla X)^b, \mathrm{Id}_{\dim_{\mathrm{loc}}}\rangle_{\mathrm{HS}} = \mathrm{tr}(\nabla X)^b$ in L^2. Since $X_i \to X$ in $H^{1,2}_H(TM)$ implies $\mathrm{div} X_i \to \mathrm{div} X$ in L^2, we conclude that $\mathrm{div} X = \mathrm{tr}(\nabla X)^b$ $\mathfrak{m}$-a.e. on E. □

We shall now use the result of Theorem 5.13 to define the N-Ricci tensor.

We start defining $\mathbf{\Gamma}_2(\cdot,\cdot) : [\mathrm{TestV}(M)]^2 \mapsto \mathrm{Meas}(M)$ by

$$\mathbf{\Gamma}_2(X,Y) := \mathbf{\Delta}\frac{\langle X,Y\rangle}{2} + \Big(\frac{1}{2}\langle X, (\Delta_{\mathrm{H}} Y^b)^\sharp\rangle + \frac{1}{2}\langle Y, (\Delta_{\mathrm{H}} X^b)^\sharp\rangle\Big)\,\mathfrak{m},$$

where $X, Y \in [\mathrm{TestV}(M)]^2$ and Δ_{H} is the Hodge Laplacian. It is proved in [23] that $\mathbf{\Gamma}_2(\nabla f, \nabla f) = \mathbf{\Gamma}_2(f)$ for $f \in \mathrm{TestF}(M)$ and that $\mathbf{\Gamma}_2(\cdot,\cdot)$ can be continuously extended to $[H^{1,2}_H(TM)]^2$. Furthermore, it is known from Theorem 3.6.7 of [23] that $\mathbf{Ricci}(X,Y) := \mathbf{\Gamma}_2(X,Y) - \langle \nabla X, \nabla Y\rangle_{\mathrm{HS}}\,\mathfrak{m}$ is a symmetric $\mathrm{TestF}(M)$-bilinear form on $[H^{1,2}_H(TM)]^2$.

We then define the measure valued map $\mathbf{R}_N$ on $[H^{1,2}_H(TM)]^2$ by

$$R_N(X,Y) := \begin{cases} \dfrac{1}{N-\dim_{\mathrm{loc}}}\big(\mathrm{tr}(\nabla X)^b - \mathrm{div}X\big)\big(\mathrm{tr}(\nabla Y)^b - \mathrm{div}Y\big) & \dim_{\mathrm{loc}} < N, \\ 0 & \dim_{\mathrm{loc}} \geq N. \end{cases}$$

From the continuity of $\mathrm{div}\cdot$ and $\mathrm{tr}(\nabla\cdot)^b$ on $H^{1,2}_H(TM)$, we deduce that $(X,Y) \mapsto R_N(X,Y)\,\mathfrak{m}$ is continuous on $[H^{1,2}_H(TM)]^2$ with values in $\mathrm{Meas}(M)$. From the calculus rules developed in [23], it is easy to see that $(X,Y) \mapsto R_N(X,Y)\,\mathfrak{m}$ is homogenous with respect to the multiplication of test functions, i.e.

$$\lambda R_N(X,Y)\,\mathfrak{m} = R_N(\lambda X, Y)\,\mathfrak{m}$$

for any $\lambda \in \mathrm{TestF}(M)$. Therefore we can define $\mathbf{Ricci}_N(\cdot,\cdot)$ on $[H^{1,2}_H(TM)]^2$ in the following way:

Definition 5.15 (Ricci tensor)**.** We define $\mathbf{Ricci}_N$ as a measure valued map on $[H^{1,2}_H(TM)]^2$ such that for any $X, Y \in H^{1,2}_H(TM)$ it holds

$$\mathbf{Ricci}_N(X,Y) \;=\; \mathbf{\Gamma}_2(X,Y) - \langle (\nabla X)^b, (\nabla Y)^b\rangle_{\mathrm{HS}}\,\mathfrak{m} - R_N(X,Y)\,\mathfrak{m}.$$

Combining the discussions above and Proposition 5.12, we know $\mathbf{Ricci}_N$ is a well defined tensor, i.e. $(X,Y) \mapsto \mathbf{Ricci}_N(X,Y)$ is a symmetric $\mathrm{TestF}(M)$-bilinear form. Then, we can prove the following theorem by combining our Theorem 5.13 and Theorem 3.6.7 of [23].

Theorem 5.16. *Let M be a* RCD$^*(K,N)$ *space. Then*

$$\mathbf{Ricci}_N(X,X) \geq K|X|^2\,\mathfrak{m},$$

and

$$\mathbf{\Gamma}_2(X,X) \quad \geq \quad \Big(\frac{(\mathrm{div}X)^2}{N} + \mathbf{Ricci}_N(X,X)\Big)\,\mathfrak{m} \tag{5.9}$$

holds for any $X \in H^{1,2}_H(TM)$. Conversely, on a RCD(K',∞) *space M, assume that*

(1) $\mathrm{dim}M \leq N$

(2) $\mathrm{tr}(\nabla X)^b = \mathrm{div}X \ \mathfrak{m} - a.e.\ on\ \{\mathrm{dim}_{\mathrm{loc}} = N\}, \forall X \in H^{1,2}_H(TM)$

(3) $\mathbf{Ricci}_N \geq K$

for some $K \in \mathbb{R}$, $N \in [1,+\infty]$, then it is RCD$^*(K,N)$.

Proof. From the definition and Proposition 5.12, we know that $\mathbf{Ricci}_N(X,X) \geq K|X|^2\,\mathfrak{m}$ means

$$\mathbf{\Gamma}_2(X,X) \geq \Big(K|X|^2 + \|(\nabla X)^b\|^2_{\mathrm{HS}} + \frac{1}{N-\mathrm{dim}_{\mathrm{loc}}}(\mathrm{tr}(\nabla X)^b - \mathrm{div}X)^2\Big)\,\mathfrak{m}. \tag{5.10}$$

Hence we need to prove (5.10) for any $X \in H^{1,2}_H(TM)$.

First of all, notice that for $X = \nabla f$, this is exactly the inequality in Theorem 5.13. Hence (5.10) holds for any $X = \nabla f$, $f \in \mathrm{TestF}(M)$.

Secondly, we need to prove (5.10) for any $X \in \mathrm{TestV}(M)$. Let $X = \sum_i g_i \nabla f_i$ be a test vector field. From the homogeneity of $R_N\,\mathfrak{m}$ and $\mathbf{\Gamma}_2(\cdot,\cdot) - \langle \nabla\cdot, \nabla\cdot\rangle_{\mathrm{HS}}\,\mathfrak{m}$ which is proved in [23] we know that $\mathbf{Ricci}_N(\cdot,\cdot)$ is a symmetric $\mathrm{TestF}(M)$-bilinear form. Therefore $\mathbf{Ricci}_N(X,X) = \sum_{i,j} g_i g_j \mathbf{Ricci}_N(\nabla f_i, \nabla f_j)$. Thus we need to prove the inequality

$$\sum_{i,j} g_i g_j \mathbf{Ricci}_N(\nabla f_i, \nabla f_j) \geq K \sum_{i,j} g_i g_j \langle \nabla f_i, \nabla f_j\rangle\,\mathfrak{m}.$$

Hence by an approximation argument, it is sufficient to prove this inequality for simple functions $g_i = \sum_{k_i=1}^{K_i} a_{i,k_i}\chi_{E_{i,k_i}}$, i.e.

$$\sum_{i,j,k_i,k_j} a_{i,k_i}a_{j,k_j}\chi_{E_{i,k_i}\cap E_{j,k_j}}\mathbf{Ricci}_N(\nabla f_i, \nabla f_j) \geq K \sum_{i,j,k_i,k_j} a_{i,k_i}a_{j,k_j}\chi_{E_{i,k_i}\cap E_{j,k_j}}\langle \nabla f_i, \nabla f_j\rangle\,\mathfrak{m}.$$

Let $E \in X$ be a Borel set with positive measure such that $E = \cap_I (E_{i,k_i} \cap E_{j,k_j})$ where $I := \{(i,j,k_i,k_j) : \mathfrak{m}(E \cap E_{i,k_i} \cap E_{j,k_j}) > 0$. We then restrict the inequality above on E

$$\sum_{(i,j,k_i,k_j)\in I} \mathbf{Ricci}_N(\nabla a_{i,k_i} f_i, \nabla a_{j,k_j} f_j)|_E \geq K \sum_{(i,j,k_i,k_j)\in I} \langle \nabla a_{i,k_i} f_i, \nabla a_{j,k_j} f_j\rangle \, \mathfrak{m}|_E,$$

which is equivalent to

$$\mathbf{Ricci}_N(\nabla F, \nabla F)|_E \geq K|\mathrm{D}F|^2 \, \mathfrak{m}|_E,$$

where $F = \sum_{(i,k_i):\exists(i,j,k_i,k_j)\in I} a_{i,k_i} f_i$. Clearly, this is true due to Theorem 5.13. Then we can repeat this argument on all E which is a decomposition of X and complete the proof.

Next, it is sufficient to prove that (5.10) can be continuously extended to $H^{1,2}_H(TM)$. It is proved in Theorem 3.6.7 of [23] that $\mathbf{\Gamma}_2(X,X) - \|\nabla X\|^2_{\mathrm{HS}} \, \mathfrak{m}$ vary continuously w.r.t. $\|\cdot\|_{\mathrm{TV}}$ as X varies in $H^{1,2}_H(M)$. The term $\frac{1}{N-\dim_{\mathrm{loc}}}(\mathrm{tr}(\nabla X)^b - \mathrm{div}X)^2 \, \mathfrak{m}$ also varies continuously in $\mathrm{Meas}(M)$ due to the property (b) and (c) of $H^{1,2}_H(TM)$. Therefore we know (5.10) holds for all $X \in H^{1,2}_H(TM)$.

Moreover, from the definition of $\mathbf{Ricci}_N$ we can see that

$$\begin{aligned}
\Big(\frac{(\mathrm{div}X)^2}{N} + \mathbf{Ricci}_N(X,X)\Big)\, \mathfrak{m} &= \mathbf{\Gamma}_2(X,X) - \|(\nabla X)^b\|_{\mathrm{HS}} \, \mathfrak{m} + \frac{(\mathrm{div}X)^2}{N} \, \mathfrak{m} \\
&- \frac{1}{N-\dim_{\mathrm{loc}}}\big(\mathrm{tr}(\nabla X)^b - \mathrm{div}X\big)^2 \, \mathfrak{m} \\
&\leq \mathbf{\Gamma}_2(X,X) - \frac{(\mathrm{tr}(\nabla X)^b)^2}{\dim_{\mathrm{loc}}} \, \mathfrak{m} + \frac{(\mathrm{div}X)^2}{N} \, \mathfrak{m} \\
&- \frac{1}{N-\dim_{\mathrm{loc}}}(\mathrm{tr}(\nabla X)^b - \mathrm{div}X)^2\Big) \, \mathfrak{m} \\
&\leq \mathbf{\Gamma}_2(X,X)
\end{aligned}$$

which is the inequality (5.9).

Conversely, picking $X = \nabla f$, $f \in \mathrm{TestF}$ in $\mathbf{Ricci}_N(X,X) \geq K|X|^2 \, \mathfrak{m}$, we have the following inequality according to the definition

$$\mathbf{\Gamma}_2(f) \quad \geq \quad \big(K|\mathrm{D}f|^2 + \|\mathrm{H}_f\|^2_{\mathrm{HS}} + \frac{1}{N-\dim_{\mathrm{loc}}}(\mathrm{tr}\mathrm{H}_f - \Delta f)^2\big) \, \mathfrak{m}.$$

Then by Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\mathbf{\Gamma}_2(f) &\geq \big(K|\mathrm{D}f|^2 + \frac{1}{n}(\mathrm{tr}\mathrm{H}_f)^2 + \frac{1}{N-\dim_{\mathrm{loc}}}(\mathrm{tr}\mathrm{H}_f - \Delta f)^2\big) \, \mathfrak{m} \\
&\geq \big(K|\mathrm{D}f|^2 + \frac{1}{N}(\Delta f)^2\big) \, \mathfrak{m}
\end{aligned}$$

for any $f \in \mathrm{TestF}(M)$. The conclusion follows Proposition 5.3. $\square$

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